UNIVERSITY OF MICHIGAN
MAR 23 1959
MATHEMATICS LIBRARY

Mathematical Reviews

Edited by

R. P. Boas, Jr. **J. L. Doob** **W. S. Massey**

S. H. Gould, ***Executive Editor***

Chandler Davis, W. Freiberger ***and*** **J. A. Zilber,** ***Editorial Consultants***

H. A. Pogorzelski, ***Editorial Associate***

Vol. 19, No. 10 November, 1958 pp. 1031-1150

TABLE OF CONTENTS

Foundations, Theory of Sets, Logic 1031

Algebra 1033
Combinatorial Analysis 1033
Linear Algebra 1033
Polynomials 1034
Partial Order, Lattices 1034
Fields, Rings 1035
Algebras 1036
Groups and Generalizations 1036
Homological Algebra 1037

Theory of Numbers 1038
General Theory of Numbers 1038
Analytic Theory of Numbers 1039
Theory of Algebraic Numbers 1040
Geometry of Numbers 1040

Analysis 1041
Functions of Real Variables 1041
Measure, Integration 1042
Functions of Complex Variables 1043
Geometric Analysis 1046
Harmonic Functions, Convex Functions . . . 1046
Special Functions 1047
Sequences, Series, Summability 1047
Approximations, Orthogonal Functions . . . 1049
Trigonometric Series and Integrals 1049
Integral Transforms 1050
Ordinary Differential Equations 1051
Partial Differential Equations 1055
Difference Equations, Functional Equations . 1062
Integral and Integrodifferential Equations . . 1062
Calculus of Variations 1062

Topological Algebraic Structures 1062
Topological Groups 1063
Lie Groups and Algebras 1064
Topological Vector Spaces 1064
Banach Spaces, Banach Algebras, Hilbert Spaces 1066

Topology 1068
General Topology 1068
Algebraic Topology 1069

Geometry 1071
Geometries, Euclidean and Other 1071
Convex Domains, Integral Geometry 1073
Differential Geometry 1074
Manifolds, Connections 1076
Complex Manifolds 1076
Algebraic Geometry 1078

Numerical Analysis 1079
Numerical Methods 1079
Computing Machines 1084

Probability 1085

Statistics 1093

Physical Applications 1099
Mechanics of Particles and Systems 1099
Statistical Thermodynamics and Mechanics . 1101
Elasticity, Plasticity 1102
Structure of Matter 1117
Fluid Mechanics, Acoustics 1117
Optics, Electromagnetic Theory, Circuits . . 1123
Classical Thermodynamics, Heat Transfer . . 1127
Quantum Mechanics 1128
Relativity 1138
Astronomy 1141
Geophysics 1144

Other Applications 1145
Economics, Management Science 1145
Programming, Resource Allocation, Games . . 1146
Biology and Sociology 1147
Information and Communication Theory . . . 1148
Control Systems 1148

History, Biography 1150

AUTHOR INDEX

Abian, S. ... 1042
Abrikosov, A. A. ... 1134
Adkins, J. E. ... 1105
Aharoni, A.-Shtrikman, S. ... 1125
Aizu, K. ... 1137
Akaike, H. ... 1096
Akhieser, A. I.-Prokhoda, I. G.-Sitenko, A. G. ... 1124
Albert, A. A. ... 1036
Albertoni, S.-Duimio, F. ... 1132
Al-Dhahir, M. W. ... 1072
Altman, M. ... 1068
Amat, G. See Maes, S.
Ambarcumyan, S. A. ... 1108, 1109
Amenzade, Yu. A. ... 1106
Andô, T. ... 1067
Andreev, N. I. ... 1149
Anonymous:
Luigi Cremona. ... 1150
Ulisse Dini. ... 1150
Vito Volterra. ... 1150
Obituary: Luigi Fantappiè. 1150
Obituary: Heinrich Scholz. 1150
Arcidiacono, G. ... 1139
Armitage, P. ... 1096
Aronszajn, N. ... 1056
Aronszajn, N.-Panitchpakdi, P. ... 1069
Atkinson, F. V. ... 1052
Atsumi, A. ... 1103
Atta, S. E. ... 1084
Auluck, F. C.-Kothari, D. S. ... 1143
Back, F. G.-Lowen, H. ... 1124
Baer, R. M. ... 1034
Bahadur, R. R. ... 1096
Baiada, E. ... 1041
Bakke, F.-Wergeland, H. ... 1138
Balasiński, W.-Mrówka, S. ... 1085
Barashenkov, V. S. ... 1134
Barford, N. C.-Reynolds, G. T. 1126
Barna, B. ... 1047
Barra, J.-R. ... 1088
Barry, P. D. ... 1044
Bartholomew, D. J. ... 1094
Barut, A. O. ... 1131
Bateman, P. T. ... 1040
Batkov, A. M. ... 1149
Batkov, A. M.-Solodovnikov, V. V. ... 1149
Batschelet, E. ... 1091
Baumol, W. J.-Wolfe, P. ... 1146
Beck, A. ... 1064
Beckers, H. L. ... 1120
Behnke, H.-Hermes, H. ... 1150
Beletsky, V. V. ... 1100
Belevitch, V. ... 1126
Bellemans, A. ... 1102
Bellert, S. ... 1051
Bellman, R. ... 1146
Belotserkovsky, O. M. ... 1121
Berezanskiĭ, Yu. M. ... 1061
Berge, C. ... 1147
Bergmann, O. ... 1138
Bergmann, P. G. See Newman, E.
Bergstein, L. ... 1124
Bergström, H. ... 1088
Bernard, M.-Y.-Hue, J. ... 1124
Bierlein, D. ... 1147
Bincer, A. M. ... 1134
Biot, M. A. ... 1113
Biot, M. A.-Willis, D. G. ... 1115
Blachman, N. M. ... 1093
Blackburn, N. ... 1037
Blackburn, W. S. ... 1106
Blaschke, W. ... 1099
Bliss, C. I.-Owen, A. R. G. ... 1096
Block, H. D. ... 1074
Bludman, S. A. ... 1137
Boas, R. P., Jr. ... 1062
Bocchieri, P.-Montaldi, E. ... 1133
Bogdanowicz, W. ... 1042
Boggio, T. ... 1046
Bolinder, E. F. ... 1126
Booth, A. D. ... 1079
Borges, R. ... 1147
Borowitz, S.-Klein, M. M. ... 1137
Bosch, J. E. ... 1031
Boxer, R.-Thaler, S. ... 1050
Boyce, W. E. ... 1114
Bragard, L. ... 1145
Brahmachary, R. L. ... 1140
Brand, L. ... 1034
Breiman, L. ... 1148
Breslin, J. P. ... 1118
de Broglie, L. ... 1131
Browder, F. E. ... 1057, 1061
Brown, J. L., Jr. ... 1049
Brown, R. L.-Fereday, F. ... 1094
Brumberg, V. A. ... 1141
Brun, V. ... 1150
Buckland, W. R. See Kendall, M. G.
Bühlmann, H. ... 1088
Bulgakov, V. N. ... 1107
Bureau, F. J. ... 1057
Buschman, R. G. ... 1051
Busulini, B. ... 1072
Butzer, P. L. ... 1049
Byhovskiĭ, E. B. ... 1057
Calabi, E. ... 1056
Camia, F. ... 1127
Campbell, L. L. ... 1098
Camps, F.-Lepropre, M. M. ... 1145
Carini, G. ... 1123
Carleson, L. ... 1047
Cederbaum, I. ... 1033
Čerkasov, I. D. ... 1091
Černý, G. G. ... 1122
Chan, Chi-fen. ... 1053
*Chandrasekhar, S. ... 1142
*Chapin, N. ... 1084
Charnock, H. ... 1099
Chattarji, P. P. ... 1105
Chaudhuri, A. K. See Jogdeo, S. S.
Chernikov, N. A. ... 1138
Chiffi, A. ... 1055
Chihara, T. S. ... 1047
Chisini, O. ... 1078
Chow, C. K. ... 1099
Chow, Tse-Sun. ... 1127
Christie, L. S. See White, H.
Chushkin, P. I. ... 1121
Čistyakov, V. P. ... 1092
Čitovič, P. A. See Savinov, G. V.
Cohen, E. G. D. ... 1117
Cohen, H.-Handelman, G. ... 1110
Colombo, G. ... 1063
*Comprehensive bibliography on operations research. ... 1146
*Conrady, A. E. ... 1123
Constantinescu, C. ... 1044
Conte, S. D. ... 1084
Conway, H. D. ... 1107
Cook, R. K. ... 1047
Corrsin, S. ... 1120
Corsten, L. C. A. ... 1095
*Cosgriff, R. L. ... 1149
da Costa, N. C. A. ... 1031
Cowan, R. D. ... 1127
Cox, D. R. ... 1094
Craddock, J. M. ... 1099
Crandall, S. H. ... 1110
Crane, R. R. ... 1147
Creasy, M. A. ... 1095
Croisot, R. ... 1037
Cuesta, N. ... 1032
Danzer, L. ... 1074
*Davenport, W. B., Jr.-Root, W. L. ... 1090
Davis, H. F. ... 1049
Davis, R. L. ... 1037
Davis, R. M. ... 1058
Davison, B. ... 1148
Davydov, A. S.-Filippov, G. F. 1135
Day, M. M. ... 1067
Debever, R. ... 1076
Dekker, T. J. ... 1068
Dekker, T. J.-de Groot, J. ... 1068
Deprit, A. ... 1065
Devidé, V. ... 1031
Devinatz, A. ... 1047
Diaz, J. B. ... 1083
Dieudonné, J. ... 1036
Dilgan, H. ... 1150
Dinculeanu, N. ... 1066
Dinghas, A. ... 1041, 1073
Dirac, P. A. M. ... 1132
Donoghue, W. F., Jr. ... 1066
Doob, J. L. ... 1090
Dorn, W. S.-Greenberg, H. J. . 1115

(*Continued on cover 3*)

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4(82) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place.

Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, 190 Hope St., Providence 6, R.I.

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DE FRANCE
DANSK MATEMATISK FORENING
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
POLSKIE TOWARZYSTWO MATEMATYCZNE
UNIÓN MATEMÁTICA ARGENTINA
INDIAN MATHEMATICAL SOCIETY
UNIONE MATEMATICA ITALIANA
THE SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

Editorial Office

MATHEMATICAL REVIEWS, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per year ($25 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation and by a contract with the Air Force Office of Scientific Research, Air Research and Development Command, United States Air Force. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Published for the American Mathematical Society by N.V. Erven P. Noordhoff, Uitgeversmaak, Groningen, the Netherlands.

Mathematical Reviews

Vol. 19, No. 10 NOVEMBER, 1958 pp. 1031-1150

FOUNDATIONS, THEORY OF SETS, LOGIC

da Costa, Newton Carneiro Affonso. The present status of the philosophy of mathematics. Soc. Parana. Mat. Anuário 3 (1956), 17–27. (Portuguese. English summary)

★**Stegmüller, Wolfgang. Das Wahrheitsproblem und die Idee der Semantik. Eine Einführung in die Theorien von A. Tarski und R. Carnap.** Springer-Verlag, Wien, 1957. x+328 pp. $7.85.

Das Buch stellt die Semantik, wie sie von Tarski [Studia Philos. 1 (1935), 261–405] und Carnap [Introduction to semantics, Harvard Univ. Press, 1942] entwickelt wurde, für den deutschen philosophischen Leser in gut lesbarer Form dar. Das Buch enthält nichts spezifisch Mathematisches, es sei daher hier der Inhalt nur in Stichworten angegeben: Antinomie des Lügners, Trennung von Objekt- und Metasprache als Lösung der Antinomie. Semantische Systeme als Kalküle mit Interpretation. Der semantische Folgerungsbegriff. Extension und Intension sprachlicher Ausdrücke. Das semantische System der Quantorenlogik. In einem ausführlichen Schlusskapitel stellt Verf. auch die bisherige Kritik an der Semantik, insbesondere die "pragmatistische" Kritik Quines, dar.

P. Lorenzen (Kiel).

Devidé, Vladimir. Eine Charakterisierung und Klassifikation der umkehrbar eindeutigen Abbildungen einer Menge in sich. Z. Math. Logik Grundlagen Math. 2 (1956), 228–232.

Theorem I. Let f be a (single-valued) mapping of a non-empty set A into itself. In order for f to be 1-1, it is necessary and sufficient that A can be partitioned into disjoint non-empty subsets $A_\lambda(\lambda \in \Lambda)$ such that the following conditions are fulfilled for each λ: (U1) $fA_\lambda \subseteq A_\lambda$. (U2) If both $P \subseteq A_\lambda$ and $fP=P$, then either $P=\emptyset$ or $P=A_\lambda$. (U3) The cardinal of $A_\lambda - fA_\lambda$ is ≤ 1. To show the sufficiency of Th. I, the author uses his earlier characterizations of the positive integers [Arch. Math. 6 (1955), 408–412; MR 17, 448] and of the integers [Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. 11 (1956), 11–15; MR 18, 270].

The partition referred to in Th. I is uniquely determined by the corresponding 1-1 mapping. Associated with the partition is a sequence of cardinals, k_α, k_β, k_{γ_n}, $n=1, 2, \cdots$, expressing the number of sets of the partition isomorphic (the 1-1 mapping f corresponding to the successor function) to the positive integers, the integers, and the integers modulo n, respectively. Thus

$$(*) \qquad \aleph_0(k_\alpha+k_\beta)+\sum_n n \cdot k_{\gamma_n} = \text{the cardinal of } A.$$

This sequence determines the 1-1 mapping f up to the isomorphism, and isomorphic mappings have partitions with the same associated sequence of cardinals. (The mappings f_1 and f_2 of A into A are isomorphic if $f_2=pf_1p^{-1}$, for some mapping p of A onto A.) For any k_α, k_β, k_{γ_n}, $n=1, 2, \cdots$, such that (*) holds, there is a 1-1 mapping f of A into itself such that this sequence is the sequence of cardinals of the partition corresponding to f.

W. W. Boone (Manchester).

Hajnal, András. On a consistency theorem connected with the generalized continuum problem. Z. Math. Logik Grundlagen Math. 2 (1956), 131–136.

Announcement of the following remarkable result (the proof of which will appear in detail in Acta Math. Acad. Sci. Hungar.): For any members, Λ, N of a certain wide class of ordinal numbers (called "absolutely definable"), if the inequality $\exp \aleph_\Lambda \geq \aleph_{\Lambda+N+1}$ is consistent with Gödel's axiom system Σ^*, then so is the equality $\exp \aleph_\Lambda = \aleph_{\Lambda+N+1}$, even if one assumes that $\exp \aleph_\mu = \aleph_{\mu+1}$ for every $\mu \geq \Lambda+N$ (here $\exp \aleph_\nu$ denotes $2^{\aleph_\nu}$ [reviewer's notation]). As an immediate corollary (specialized to the case $\Lambda=0$, $N=1$): the equation $\exp \aleph_0 = \aleph_1$ is demonstrable in Σ^* if and only if the inequality $\exp \aleph_0 \neq \aleph_2$, or the inequality $\exp \aleph_0 \neq \exp \aleph_1$, is demonstrable in Σ^*. It is pointed out that the proofs are constructive, so that, e.g., from a given proof that $\exp \aleph_0 \neq \aleph_2$, one can construct a proof that $\exp \aleph_0 = \aleph_1$. The outline of the construction of the model is as follows. Let λ, ν satisfy $\exp \aleph_\lambda \geq \aleph_\mu$, where [reviewer's notation] $\mu=\lambda+\nu+1$. Define a one-one mapping h on ω_μ into the power set of ω_λ, and a function k on ω_μ that associates with every $\alpha < \omega_\mu$ a one-one mapping of α onto its cardinal. Define S, J, K_1, K_2 and J_i as in Gödel's construction of Δ, but this time with respect to the class $12 \times On^2$ (instead of $9 \times On^2$). The model is that determined by $\mathfrak{B}(G)$, where G is the following function on On. Write $\beta=K_1{}'\alpha$, $\gamma=K_2{}'\alpha$, and [reviewer's notation] $W_i=\mathfrak{B}(J_i)$. The definition of G on W_i $(i<9)$ is like that of F: $G'\alpha=G''\alpha$ for $\alpha \in W_0$, and $F_i(G'\beta, G'\gamma)$ for $\alpha \in W_i$ $(0<i<9)$. On the other classes its values are as follows. For $\alpha < \omega_\mu$: $G'\alpha = G'\beta \cdot On$ $(\alpha \in W_9)$, $G'\beta \cdot k'G'\gamma$ $(\alpha \in W_{10})$, $G'\beta \cdot k'G'\gamma$ $(\alpha \in W_{11})$; and for $\alpha \geq \omega_\mu$: $G'\alpha=0$ $(\alpha \in W_9)$, $G'\beta \cdot h$ $(\alpha \in W_{10})$, $G'\beta \cdot k$ $(\alpha \in W_{11})$.

L. Gillman (Lafayette, Ind.).

Nikodým, Otton Martin; and Nikodým, Stanisława (Mrs. Otton Martin). Some theorems on divisibility of infinite cardinals. Arch. Math. 8 (1957), 96–103.

The authors' definition of divisibility amounts to this: $\aleph_\alpha | \aleph_\beta$ means that ω_β is cofinal with ω_α [cf. Bachmann, Transfinite Zahlen, Springer, Berlin, 1955, p. 24; MR 17, 134]. Theorems 6 and 7 are false (for a counterexample, take $p=\aleph_\omega$, $q=\aleph_{\omega+\omega}$); the rest follow readily from the notions of initial and regular ordinals and cofinality — e.g., if $\aleph_\alpha | \aleph_\beta$ and $\aleph_\alpha | \aleph_\gamma$, then either $\aleph_\beta | \aleph_\gamma$ or $\aleph_\gamma | \aleph_\beta$.

F. Bagemihl (Notre Dame, Ind.).

Bosch, Jorge E. Fixed points of transfinite ordinal operators. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. 5 (1956), 201–214 (1957). (Spanish)

Let W denote the class of ordinal numbers, and let $\varphi(\xi) \in W$ for every $\xi \in W$. Conditions on the function φ

are given for the existence of a subclass F of W, cofinal with W, such that $\varphi(\xi)=\xi$ for every $\xi \in F$. In particular, the assumptions in the theorem on the existence of critical numbers of normal functions [cf. Bachmann, Transfinite Zahlen, Springer, Berlin, 1955, p. 38; MR **17**, 134] are weakened. The existence of fixed points for certain transformations of the Cartesian power W^σ (where σ is an ordinal number) into itself is also proved.

F. Bagemihl (Notre Dame, Ind.).

Uspenskij, V. A. Some notes on recursively enumerable sets. Z. Math. Logik Grundlagen Math. **3** (1957), 157–170. (Russian. English summary)

The author discusses questions which concern relations among recursively enumerable sets, calculable functions, and calculable numberings, in particular, Gödel numbering. The calculable functions of greatest interest are those whose domains are natural numbers and whose ranges are i-tuples of natural numbers. It is shown in Section 3 that there is no calculable numbering of the system of all infinite recursively enumerable sets of natural numbers. Many of the notions considered arise in Post, Bull. Amer. Math. Soc. **50** (1944), 284–316 [MR **6**, 29], and Dekker, Proc. Amer. Math. Soc. **4** (1953), 495–501 [MR **15**, 385]. An example of a recursively enumerable set of natural number pairs is furnished in Section 4 whose set of lower points is not recursively enumerable. A revision of the Post-Dekker classification of infinite sets is introduced in Section 5 and its relation to the original classification considered. In Section 6, the following theorem concerning hyper-immune sets is established: A direct enumeration of an infinite set M cannot be majorized by any computable function defined on the natural number sequence if and only if M is a hyper-immune set. Errata: Page 158, line 5, m_1 for m_2; page 159, line 2, x for z; page 165, line −12, M for n.

E. J. Cogan (Bronxville, N.Y.).

Montague, Richard; and Kalish, Donald. Remarks on descriptions and natural deduction. II. Arch. Math. Logik Grundlagenforsch. **3** (1957), 65–73.

[For part I see same Arch. **3** (1957), 50–64; MR **19**, 724.] In this second part the authors present a list of eight rules for natural deduction in an abstract language L of a general nature. A natural deduction is a column of lines in which the first has the form 'Show φ' where φ is a formula of L to be proved, and each subsequent line is either a formula of L, or has the form 'Show ψ' where ψ is a formula of L. Eight rules provide for the cancellation of the word 'Show' in a given line and the marking of the lines below it under certain circumstances. The formula φ is said to be proved if 'Show φ' is the first line of a deduction, the word 'Show' in that line is cancelled, and the lines below the first line are marked. This system seems closely related to the work of Gentzen [Math. Z. **39** (1934), 176–210, 405–431] both in the form of the rules and in the fact that the final form of a proof in the authors' sense can be exhibited as a tree. It is well known that the class of theorems determined by the first six rules are the theorems of the pure first order predicate calculus, and that the first seven rules determine the class of theorems of the first order predicate calculus with identity. The great part of the paper is devoted to a proof that the eight rules together determine the class of theorems of the language L, developed in the first part of the paper, of the predicate calculus with equality and descriptions. Errata: The last two lines in the second column deduction of Footnote 19 should be marked. The first four lines of page 66 should occur at the bottom of the page as a continuation of the footnote on page 65.

E. J. Cogan.

Cuesta, N. A deductive science with separate irreducible bases. Collect. Math. **8** (1955–1956), 73–84. (Spanish)

Within a certain deductive structure [see Cuesta, Rev. Mat. Hisp.-Amer. (4) **14** (1954), 104–117; MR **16**, 987], the author constructs a mathematical theory (founded on a set A of propositions) which, if only bases contained in A are considered, has no postulates and has ten irreducible bases of three propositions each, and none of these propositions is an element of every one of these bases.

F. Bagemihl (Notre Dame, Ind.).

Rasiowa, H. Errata to the paper "On the ε-theorems". Fund. Math. **44** (1957), 333.

A short list of corrections to the paper in Fund. Math. **43** (1956), 156–165 [MR **18**, 711].

Schneider, H. Logische Symbole in der mathematischen Literatur. Math.-Phys. Semesterber. **5** (1957), 252–260.

This paper begins with a discussion of the justifications for introducing logical notation in the study of mathematics and science. Truth tables and some equivalences from the calculus of propositions are sketched. The use of quantifiers is illustrated by providing a number of equivalent logical formulations of the negation of the statement that a function is uniformly continuous in an interval.

E. J. Cogan (Bronxville, N.Y.).

Robinson, A. Some problems of definability in the lower predicate calculus. Fund. Math. **44** (1957), 309–329.

The author is motivated by two related algebraic problems to investigate further the theory of models. In particular, results concerning relative model-consistency and completeness of two sets of statements are obtained from which the algebraic results are derived. Much of the terminology is taken from the author's "Complete theories" [North-Holland Publ. Co., Amsterdam, 1956; MR **17**, 817]; and this paper extends his results in Math. Ann. **130** (1956), 405–409 [MR **17**, 1180]. The algebraic results obtained are these. (1) If M is an ordered field such that there is a uniform bound for the number of squares needed to represent each positive element of M as a sum of squares then the same is true of any finite algebraic extension M' of M. (2) Let R be the field of rationals and R^* the field of real numbers, and let $Q^*(y_1, \cdots, y_n)$ be a predicate of rationals $y_1, \cdots, y_n$ in the lower predicate calculus with addition, multiplication, equality, and order. Then there is a predicate $Q(y_1, \cdots, y_n)$ which holds in R whenever $Q^*(y_1, \cdots, y_n)$ holds in R^* for rational R. The second of these is formulated and proved in a more general way in the theory of models.

E. J. Cogan.

See also: Groups and Generalizations: Levi. **General Topology:** Dekker and de Groot; Dekker ;Viola; Mrówka; de Vries. **Biology and Sociology:** Dunham.

ALGEBRA

★**Miller, Kenneth S. Elements of modern abstract algebra.** Harper's Mathematics Series. Harper & Brothers, New York, 1958. vii+188 pp. $5.00.

In the opinion of the reviewer, this is a poorly written textbook, covering the usual basic material on groups, rings, and fields. The book is not at all affected by the changes in modern algebra in the last twenty-eight years and the reviewer fails to see how any student could benefit by its use. The proofs are often needlessly long and complicated (e.g., the construction of a quotient field of an integral domain occupies 3½ printed pages) and the book is burdened with non-illuminating examples. For the most part, the exercises are routine, except for one or two which are much too difficult for the intended audience: Ex. 3.51 on p. 167 asks for a proof that every finite division ring is commutative, with no hints whatsoever being given!

There are far too many criticisms of exposition, arrangement, and choice of material that the reviewer feels should be made to warrant printing them here. The following samples should suffice: 1. It is surely out of place nowadays to prove the fundamental theorem for finite abelian groups, p. 46, without studying modules over Euclidean or principal ideal rings. This is done in spite of the fact that there is a long discussion of Euclidean rings and unique factorization. 2. Starting on p. 67, equivalence relations and classes are treated in ponderous detail, yet these concepts are never mentioned, let alone used, in connection with cosets or residue classes. 3. The chapter on rings has a proof of the Hilbert basis theorem, and the field chapter includes an incomplete treatment of finite fields. Now, the Hilbert theorem is pointless without further work showing its application to ideals in polynomial rings and algebraic geometry, while a treatment of finite fields that fails to state that a finite field is determined up to isomorphism by the number of its elements and does not touch the existence theorem is not useful. *A. Rosenberg* (Evanston, Ill.).

Thurston, H. A. Derived operations and congruences. Proc. London Math. Soc. (3) **8** (1958), 127–134.

A section of a continued product of operations of an algebra A is called a derived operation of A. A pair of ternary operations α and β is biternary if $\beta\alpha xyzyz=x$, $\alpha\beta xyzyz=x$ and $\alpha xxz=z$ for every x, y, and z of A; a pair of quaternary operations γ and δ is biquaternary if $\gamma xxyz=z$, $\delta xyzx=y$ and $\delta\gamma xyzxyzx=x$ for every x, y, z, of A. The author proves that any algebra with derived biternary operations has derived quaternary operations. This and similar results are used to extend results of A. I. Mal'cev [Mat. Sb. N.S. **35(77)** (1954), 3–20; MR **16**, 440] on the relationship between the properties of the semigroup Σ generated by singulary sections and such properties as normality, properness, the existence of a derived ternary operation ψ satisfying $\psi xxy=\psi yxx=y$, and the existence of the above types of derived operations. The results of Mal'cev and the author are summarized in a Hasse diagram. *A. A. Grau* (Oak Ridge, Tenn.).

Combinatorial Analysis

Lohne, Johannes. Graphical illustration of summation formulae. II. Nordisk Mat. Tidskr. **5** (1957), 136–138, 168. (Norwegian. English summary)

The well-known formula

$$\sum_{x=1}^{n}\binom{p+x-1}{p}=\binom{n+p}{p+1}$$

is illustrated graphically for $p=2$, 3, and a geometric representation of Pascal's triangle for binomial coefficients is given. *From the author's summary.*

See also: Groups and Generalizations: Yacoub. **Geometry of Numbers:** Stöhr.

Linear Algebra

★**Kochendörffer, R. Determinanten und Matrizen.** Mathematisch-Naturwissenschaftliche Bibliothek, 12. B. G. Teubner Verlagsgesellschaft, Leipzig, 1957. vi+144 pp. DM 6.60.

This is a book for beginners, but tries to point the way to more advanced work. It starts with determinants and matrices (nowadays it is more fashionable to introduce determinants subsequent to matrices) giving Weierstrass' characterization of determinants. Then vector spaces are introduced. The next chapter deals with linear systems and includes a section on numerical methods, namely elimination and two standard iteration processes. Although symmetric matrices are only introduced systematically in the next chapter their importance in the convergence of one of these processes is studied. Next special topics like Hadamard's inequality, Schur's lemma, the Kronecker product, commutative matrices and their eigenvalues and similarity to triangular matrices are studied. Then comes a chapter on similarities, including the Jordan normal form where the author follows the treatment of Reichardt [Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **3** (1954), 445–447; MR **16**, 894]. *O. Taussky-Todd.*

Jacob, H. G. Coherence invariant mappings of symmetric transformations. Proc. Amer. Math. Soc. **8** (1957), 943–949.

Let $\mathfrak{X}$ be a vector space of dimension $\geqq 3$ over a field Φ of characteristic $\neq 2$. Let $\mathfrak{X}$ be self-dual relative to a non-degenerate hermitian scalar product with associated involution $\alpha\to\alpha^*$ and $\mathfrak{S}$ be the set of symmetric transformations on $\mathfrak{X}$ of finite rank. If σ is a coherence invariant mapping of $\mathfrak{S}$ onto itself there exists a nonsingular semi-linear transformation S on $\mathfrak{X}$ with the associated automorphism τ and a self-adjoint element $\lambda=\lambda^*\in\Phi$ such that $T^\sigma=T[S\lambda_\Phi\otimes S]$, for all $T\in\mathfrak{S}$. Moreover, $\gamma^{*\tau}=\gamma^{\tau*}$ for $\gamma\in\Phi$. This is a generalization of the reviewer's result which restricts $\alpha\to\alpha^*$ to be the identity mapping and the dimension of $\mathfrak{X}$ to be finite. *L. K. Hua* (Peking).

Cederbaum, I. Matrices all of whose elements and subdeterminants are 1, −1 or 0. J. Math. Phys. **36** (1958), 351–361.

A square or rectangular matrix with the property of the title is called by the author an E-matrix. A necessary and sufficient condition that a square matrix A of order n be an E-matrix is that, assuming any $n-1$ of the $2n$ elements of the n-rowed column vectors X, Y to be zero, X and Y can be found such that $Y=AX$, $X\neq 0$, the remaining $n+1$ unspecified elements of X, Y being each $+1$, -1, or 0. This criterion leads to the result that if $\begin{bmatrix}A_{11}, A_{12}\\ A_{21}, A_{22}\end{bmatrix}$ is an

E-matrix with $|A_{22}|\neq 0$, then $\begin{bmatrix} A_{11}-A_{12}A_{22}^{-1}A_{21}, & A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}, & A_{22}^{-1} \end{bmatrix}$ is also an E-matrix.

The author further describes the structural properties of non-singular E-matrices and discusses congruent transformations by E-matrices. *D. E. Rutherford.*

Brand, Louis. The Pi theorem of dimensional analysis. Arch. Rational Mech. Anal. **1** (1957), 35–45.

A very clear version of the essentially known, purely algebraic proof of the Pi Theorem, together with a brief discussion of related proofs. *G. Birkhoff.*

Herz, Jean-Claude. Sur la démonstration du théorème de réduction de Wielandt dans le cas fini. Chiffres **1** (1958), 23–24.

In Math. Z. **50** (1944), 93–143 [MR **8**, 157], Wielandt has shown that if λ is an eigenvalue, x a corresponding eigenvector of the real or complex matrix A and v a column vector such that $v^*x=1$ (where $v^*=\bar{v}'$), then the matrix $A_1=A-xv^*$ has the same eigenvalues as A, except that the multiplicity of λ has decreased by one and that of 0 has increased by one: $\zeta|\zeta I-A|=(\zeta-\lambda)|\zeta I-A_1|$. The author proves the fact by making use of a triangular matrix U^*AU obtained from A by a unitary transformation U. *H. Schwerdtfeger* (Montreal, P.Q.).

Mirsky, L. Matrices with prescribed characteristic roots and diagonal elements. J. London Math. Soc. **33** (1958), 14–21.

It is shown that any n real or complex numbers $w_1, \cdots, w_n$ can be the characteristic roots of a matrix A whose diagonal elements $a_1, \cdots, a_n$ are also preassigned if and only if $\sum w_i = \sum a_i$. The conditions for the case where the w_i and a_i are real and A is real symmetric were previously studied by A. Horn [Amer. J. Math. **76** (1954), 620–630; MR **16**, 105] by means of the Hardy-Littlewood-Pólya theorem [see also K. Fan, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 652–655; MR **11**, 600]. A new proof is given here which also implies the latter theorem. Another consequence is a set of conditions for n complex numbers $w_1, \cdots, w_n$ to be the characteristic roots of a matrix A if the real characteristic roots $\alpha_1, \cdots, \alpha_n$ of $\frac{1}{2}(A+A^*)$ are also preassigned [see also K. Fan, ibid. **36** (1950), 31–35; MR **11**, 526]. *O. Taussky-Todd* (Pasadena, Calif.).

Ostrowski, A. M. Note on bounds for some determinants. Duke Math. J. **22**, 95–102 (1955).

The author improves results of E. V. Haynsworth [same J. **20** (1953), 199–209; MR **14**, 837]. The following two theorems are typical. For an nth order matrix $A=(a_{\mu\nu})$ with complex elements suppose

$$\alpha_\mu=\min_\nu|a_{\mu\nu}|,\ \delta_\mu=|a_{\mu\mu}|-\sum_{\nu\neq\mu}|a_{\mu\nu}|\geqq 0,\ \alpha_\mu+\delta_\mu>0$$

for $\mu=1, \cdots, n$; then

$$|\det A|\geqq\left[1-\sum_{\nu=1}^{n}\frac{\alpha_\nu}{\delta_\nu+n\alpha_\nu}\right]\prod_{\nu=1}^{n}(\delta_\nu+n\alpha_\nu).$$

Also, if the elements of A are real and

$$nA_\mu\leqq\sum_{\nu=1}^{n}a_{\mu\nu}\leqq 2a_{\mu\mu}+(n-2)a_\mu\ \ (\mu=1, \cdots, n),$$

where A_μ and a_μ are the maximum and minimum of all nondiagonal elements in the μth row, then A is semi-monotonic, i.e., for $\kappa=1, \cdots, n$,

$$\Delta_\kappa=\det(a_{\mu\nu}-a_{\mu\kappa})\geqq 0\ \ (\mu, \nu=1, \cdots, n;\ \mu, \nu\neq\kappa).$$

This entails $\det(a_{\mu\nu}+c_\mu)\geqq\det(a_{\mu\nu})$ whenever $c_\mu\geqq 0$ $(\mu=1, \cdots, n)$. *H. S. A. Potter* (Aberdeen).

Tenca, L. Su una classe di matrici infinite. Period. Mat. (4) **35** (1957), 219–223.

Vandermonde's determinant formula is applied to evaluate determinants of finite submatrices of the matrix with the elements $a_{mn}=(a+nd)^m$ $(m, n=\cdots, -2, -1, 0, 1, 2, \cdots)$. The sequences of resulting determinants are studied for n taken as consecutive terms of an arithmetic progression, and for various ranges of consecutive integers m. Also the ordinary powers x^m are replaced by symbolical powers $x^{\overline{m}}=x(x+1)\cdots(x+m-1)$. Finally, referring to the symbolical notation of the Bernoulli numbers, the author evaluates the determinant

$$\left|\frac{(h+n+B)^{m+1}-(h+B)^{m+1}}{m+1}\right|$$

$(h=a, a+1, \cdots, a+n-1, m=0, 1, \cdots, n-1)$, for which he finds the value $(2!3!\cdots(n-1)!)^2$, whatever a may be, and also evaluates two determinants of similar type. *H. Schwerdtfeger* (Montreal, Q.).

Varini, Bruno. Aspetto tensoriale della teoria dei determinanti. Archimede **9** (1957), 97–104.

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: Easterfield. **Numerical Methods:** Stojaković; Hestenes; Mitra.

Polynomials

Gasapina, U. Il teorema fondamentale dell'algebra. Period. Mat. (4) **35** (1957), 149–163.

Elementary expository article. Stress is laid on the wide variety of methods that have been used in proving the fundamental theorem of algebra. Sketches of various proofs and brief historical notes are presented. *Bernard Epstein* (Philadelphia, Pa.).

Partial Order, Lattices

Baer, R. M. Certain homomorphisms onto chains. Arch. Math. **8** (1957), 93–95.

If D is a chain in a partially ordered set X, then a cut in D is a decomposition (A, B) of D into two disjoint sub-sets A, B such that $a\in A$, $b\in B$ implies that $a<b$; such a cut is called a gap if A has no last element and B has no first element. An element $x\in X$ is called an interior point of the gap (A, B) if $a\in A$, $b\in B$ implies that $a<x<b$. The following is a slightly simplified version of the author's result: Let X be a partially ordered set and D a chain in X which is isomorphic to a chain C. Then the isomorphism $\eta: D\to C$ can be extended to a homomorphism of X onto C if and only if every gap of C has no interior points. *J. Hartmanis* (Columbus, Ohio).

Dwinger, Ph. Direct sums and direct products in completely modular complete lattices. Arch. Math. **8** (1957), 85–92.

The author's results [Nederl. Akad. Wetensch. Proc. Ser. A **59** (1956), 435–443; MR **18**, 186] on direct sums and products in modular complete lattices satisfying an axiom of R. Baer are generalized to ones satisfying a weaker

axiom of A. G. Kurosch: $(\sum_{\alpha\in A} x_\alpha)\prod_{\alpha\in A} y_\alpha=\sum_{\alpha\in A} x_\alpha y_\alpha$ if $x_\alpha\leqq y_\beta$, $\alpha\neq\beta$, $\alpha\in A$, $\beta\in A$, for all index sets A. If $a=\sum_{\alpha\in A} x_\alpha$, $|A|>1$, then a is called the direct sum of $\{x_\alpha\}$, denoted $a=\sum'_{\alpha\in A} x_\alpha$, if, for every $\alpha\in A$, $x_\alpha\sum_{\beta\in A,\beta\neq\alpha} x_\beta=0$. Direct products are defined dually, but since Kurosch's axiom is not self-dual, one considers regular direct products, those in which, if $a=\prod'_{\alpha\in A} x_\alpha$, then for every $A_1\subset A$, $\prod_{\alpha\in A} x_\alpha+\prod_{\alpha\in A-A_1} x_\alpha=I$; this is the dual of a basic property of direct sums, and any finite direct product is regular. Properties of representations of I as such a sum and O as such a product are studied; in particular, in a complete lattice satisfying Kurosch's axiom (that is, a completely modular complete lattice), there is an explicit one-to-one correspondence between representations of I as a direct sum and O as a regular direct product. The lattices of normal subgroups and of congruence relations of a group satisfy Kurosch's axiom, and the representation theorems apply accordingly.

P. M. Whitman (Silver Spring, Md.).

Nakamura, Masahiro. The permutability in a certain orthocomplemented lattice. Kōdai Math. Sem. Rep. **9** (1957), 158–160.

Two elements a and b of an orthocomplemented lattice are called permutable if $a=(a\cap b)\cup(a\cap b')$. Theorems: Permutability is symmetric if and only if $a\leqq b$ implies $b=a\cup(a'\cap b)$; and lattices in which this holds for all a, b with $a\leqq b$ are called symmetric lattices. In a symmetric lattice L, a unary operation $x\to x^*$ into L has the form $x\to x^a=(x\cup a')\cap a$, called a "Sasaki projection", if and only if it is idempotent, join-endomorphic, and carries I/a' isomorphically onto a/O. Hence, in a symmetric lattice, elements are permutable if and only if the corresponding projections are permutable. *P. M. Whitman.*

See also: Groups and Generalizations: Mitrinovitch.

Fields, Rings

Yamanashi, Shin-ichi. Differentials in algebraic function field. Sci. Rep. Saitama Univ. Ser. A. **2** (1957), 125–135.

This is essentially a rearrangement of the development of the theory of differentials in function fields as given by Chevalley [Introduction to the theory of algebraic functions of one variable, Math. Surveys, no. 6, Amer. Math. Soc., New York, 1951; MR **13**, 64]. The main difference is that the theory is first given for an algebraically closed field K of constants, with the advantage that the exact differential $(dx)_{K(x)}$ is then more easily definable explicitly as a linear function on the repartitions. R is a separably generated function field with an arbitrary field of constants F. The author first extends R to $R(K)$, where K is an algebraically closed field containing F; and then defines $(dx)_R$ as $(dx)_{R(K)}\circ\mathrm{Cosp}_{R/R(K)}$, where $\mathrm{Cosp}_{R/R(K)}$ is the cotrace map of repartitions of R/F to repartitions of $R(K)/K$, as defined by Chevalley. The justification of this definition resides in Theorems 10 and 11, p. 119, of Chevalley's book. *G. P. Hochschild.*

Sorkin, Yu. I. Rings as sets with one operation subject to a unique identity. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 357–362. (Russian)

The author extends to (not necessarily associative) rings the description by Higman and Neumann [Publ. Math. Debrecen **2** (1952), 215–221; MR **15**, 284] of groups as sets with a single binary operation and a universal identity. He first points out that no one binary operation with any finite number of identities can define all rings. However the ternary operation $\nabla(x, y, z)=xz-yz+x-y$, with a suitable universal identity, serves to define an arbitrary class of rings that satisfy a finite number of identities. *E. Hewitt.*

Herstein, I. N. A condition for the commutativity of rings. Canad. J. Math. **9** (1957), 583–586.

L'auteur donne ici la condition nécessaire et suffisante (C) pour qu'un anneau donné A soit commutatif en partant de la condition suffisante connue de N. Jacobson [Ann. of Math. (2) **46** (1945), 695–707; MR **7**, 238]. Cette condition (C) est que pour tous x, y de A il existe un n entier >1 avec $(xy-yx)^n=xy-yx$. On se ramène aux cas particuliers où A est successivement un anneau de division, un anneau primitif et un anneau semi-simple. Le premier cas se traite en considérant le centre de A qui est alors algébrique sur son sous-corps premier dès qu'il existe un commutateur non nul; les autres s'étudient en montrant que tout idempotent est dans le centre de A.

J. Guerindon (Paris).

Smiley, M. F. Kleinfeld's proof of the Bruck-Kleinfeld-Skornjakov theorem. Math. Ann. **134** (1957), 53–57.

This paper combines some later results by Kleinfeld [Ann. of Math. (2) **58** (1953), 544–547; MR **15**, 392] to give a direct proof of the theorem that every alternative division ring is either associative or is a Cayley-Dickson algebra over its center, proved originally by Bruck and Kleinfeld [Proc. Amer. Math. Soc. **2** (1951), 878–890; MR **13**, 526] and independently by Skornyakov [Ukrain. Mat. Ž. **2** (1950), no. 1, 70–85; MR **12**, 668].

Marshall Hall, Jr. (Columbus, Ohio).

Maranda, J.-M. Factorization rings. Canad. J. Math. **9** (1957), 597–623.

The author shows how W. Krull's theory of principal orders and Dedekind rings in fields [Math. Ann. **105** (1931), 1–14] can be generalized to arbitrary commutative rings with unit. Thus, he develops an extension of the Artin-van der Waerden theory of quasi-divisibility, and an adaptation of the methods of valuation theory to his rings by replacing the classical partially ordered value groups by semi-groups. A typical result states that the ascending chain condition for ideals or in a generalized Dedekind ring (quasi equality of ideals containing regular elements implies equality and unique factorization assumed) implies $\mathfrak{a}=\mathfrak{a}_S\prod \mathfrak{p}^m$ where $\mathfrak{a}_S$ is the component of $\mathfrak{a}$ with respect to the set of regular elements S of the ring $\mathfrak{O}$ [in the sense of D. G. Northcott, Ideal theory, Cambridge, 1953; MR **15**, 390] and where $\{\mathfrak{p}\}$ is a finite set of prime ideals determined by valuations of $\mathfrak{O}$ in the generalized sense. *O. F. G. Schilling* (Chicago, Ill.).

Ribenboim, P. Le théorème d'approximation pour les valuations de Krull. Math. Z. **68** (1957), 1–18.

Some of the more important tools in the theory of algebraic numbers and the arithmetic theory of covering varieties of algebraic varieties etc., are furnished by appropriate generalizations of the Chinese Remainder Theorem. W. Krull developed one of these extensions for finite sets of valuations w_i of a field K, $1\leqq i\leqq r$, for which there are given elements $b_i\in K$ and α_i in the corresponding value groups Γ_i. He proved the existence of an element $x\in K$ satisfying simultaneously $w_i(x-b_i)=\alpha_i$, provided

the given (uncomparable) valuations are independent in the sense that no two of the associated valuation rings have a common non-trivial prime ideal. The author shows in this paper how Krull's theorem can be extended to arbitrary finite sets of valuations, provided certain conditions are imposed on the α_i and the values $w_i(b_i-b_j)$. Suppose Δ_{ij} $[\Delta_{ji}]$ is the isolated subgroup of Γ_i $[\Gamma_j]$ which corresponds to the largest prime ideal $\mathfrak{p}$ common to the valuation rings A_i and A_j of w_i and w_j, respectively. Then Krull proved that the order isomorphic groups Γ_i/Δ_{ij} and Γ_j/Δ_{ji} represent, if identified in an obvious fashion, the value group of the valuation ring $(A_i)_\mathfrak{p}=(A_j)_\mathfrak{p}$.

Denote now by θ_{ij} $[\theta_{ji}]$ the canonical homomorphism $\Gamma_i\to\Gamma_i/\Delta_{ij}$ $[\Gamma_j\to\Gamma_j/\Delta_{ji}]$; then a set $(\alpha_1, \cdots, \alpha_r)$ of values $\alpha_i\in\Gamma_i$ is called compatible if $\theta_{ij}(\alpha_i)=\theta_{ji}(\alpha_j)$ for $i\neq j$. One of the author's definitive results (which has numerous applications to ring theory and the Hilbert theory for valuations) states: Assuming that the given values $\alpha_i\in\Gamma_i$ are compatible [this is not a necessary condition], the system of equations $w_i(x-b_i)=\alpha_i$ has a solution $x\in K$ if and only if $\alpha_i-w_i(b_i-b_j)\in\Delta_{ij}$, provided $w_i(b_i-b_j)<\alpha_i$ for the given elements b_i. It is first shown that it suffices to prove the theorem assuming $w_i(b_i)<\alpha_i$; then a clever induction, on the length of a tree of valuations determined by the w_i, is applied. This tree is defined as follows: If w and w' are valuations of K, define the ordering $w\leqq w'$ to hold if $A\supseteq A'$ for the corresponding valuation rings; the lower bound $w\wedge w'$ is the valuation belonging to the valuation ring $A_\mathfrak{p}=A'_\mathfrak{p}$, where $\mathfrak{p}$ is the longest common prime ideal of A and A'. Thus, independence of w and w' means that $w\wedge w'$ is the trivial valuation. Suppose now that w_i are such that never $A_i\subseteq A_j$ if $i\neq j$; i.e., they are incomparable with respect to $\leqq$. Then the tree, ordered by $\leqq$, defined by $w_1, \cdots, w_r$, comprises all valuations of the form $w_{i_1}\wedge\cdots\wedge w_{i_t}$ $(1\leqq i_1<\cdots<i_t<r)$. The length of this tree is defined as maximal length of its chains. The intricate details of proof are then carried out with modification of elements and values as developed by Krull. *O. F. G. Schilling* (Chicago, Ill.).

See also: Algebra: Miller.

Algebras

Dieudonné, Jean. Sur le polynôme principal d'une algèbre. Arch. Math. **8** (1957), 81–84.

Suppose that A is an associative algebra of finite rank over the field K whose radical R has the precise exponent k. Furthermore, let B_i, $1\leqq i\leqq q$, be the simple components of A/R. Now take a K-basis of A which is made up from a basis of R and one of some subspace of A/K which is supplementary to R such that its residue classes modulo R form bases of the B_i/K. Assume that P_i, $1\leqq i\leqq q$, are the principal polynomials of the algebras B_i/K with respect to the latter bases, and P the principal polynomial for the composite basis of A/K. The author gives an elegant proof for the equation $P=\prod_{i=1}^{q}P_i^{k_i}$, $1\leqq k_i\leqq k$, which seems to have been missed in the now-classical literature on this subject. *O. F. G. Schilling.*

Albert, A. A. A construction of exceptional Jordan division algebras. Ann. of Math. (2) **67** (1958), 1–28.

Over an algebraically closed base field of characteristic $\neq 2$, the Jordan algebra M_3^8 of 3×3 hermitian matrices with Cayley number coefficients is the only simple finite-dimensional Jordan algebra which is exceptional, i.e., cannot be obtained by introducing the product $\frac{1}{2}(ab+ba)$ in an associative algebra. The present paper is concerned with exceptional central simple finite-dimensional Jordan algebras over an arbitrary base field, i.e., those algebras which become M_3^8 when the base field is extended to its algebraic closure. Such an algebra is called reduced if it contains 3 mutually orthogonal idempotents; reduced algebras therefore contain zero divisors, and in fact, are algebras of 3×3 (generalized) hermitian matrices with coefficients in a Cayley algebra [they are described by the author and N. Jacobson in Ann. of Math. (2) **66** (1957), 400–417; MR **19**, 527]. Schafer erroneously stated [Amer. J. Math. **70** (1948), 82–94; MR **9**, 408] that all exceptional algebras are reduced (his error, according to the author, being the assertion of existence of a subalgebra of dimension 15). The author constructs algebras which are not reduced and, in fact, are division algebras in the strong sense that they contain no zero divisors. [A more inclusive definition of division algebras may be found in Jacobson, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 140–147; MR **17**, 822.]

The main results are: an exceptional central simple algebra is either reduced or is a division algebra. The algebra is reduced if and only if the algebra obtained by a quadratic extension of the base field is reduced. After an extension of degree one or two of the base field, a division algebra becomes cyclic, i.e., contains a cyclic cubic field over the center. The structure of division algebras is described in terms of the cyclic cubic fields. If F is any field of characteristic not 2 over which there exists an associative cyclic nine-dimensional division algebra, then there exists an exceptional Jordan division algebra whose center is an extension of F. All algebras over finite fields, algebraic number fields, and real closed fields, are reduced. *B. Harris* (Evanston, Ill.).

Groups and Generalizations

Levi, F. W. Darstellung der Komposition in einer Gruppe als Relation. Arch. Math. **8** (1957), 169–170.

The following postulates concerning the set $\mathfrak{M}$ with relation R: $[a, b, c]$ serve to define a group.

I. There exists a $0\in\mathfrak{M}$ such that $[0, 0, 0,]\in R$: Existence of neutral element.

II. If $a, b\in\mathfrak{M}$, then there exists a $c\in\mathfrak{M}$ such that $[a, b, c]\in R$: Closure.

II′ [If $a\in\mathfrak{M}$, then there exists an $a'\in\mathfrak{M}$ such that $[a, 0, a']\in R$: Existence of inverse.

III. If $[a, 0, b]\in R$, then $b=a'$; if $[b, 0, a']\in R$, then $b=a$: Uniqueness of inverse.

IV. If $[a, b, c]\in R$, then $[b, c, a]\in R$: Cyclic property.

V. If $[a, b, c,]\in R$, then $[c', b', a']\in R$: Anticyclic property of inverse.

VI. If $[a, b', \gamma']\in R$, $[b, c', \alpha']\in R$, $[c, a', \beta']\in R$, then $[\alpha, \beta, \gamma]\in R$: Analog of associative law.

I, II′, III, IV, V, VI together yield VII: $[a, b, c]\in R$ determines c uniquely from a, b: Uniqueness. The author shows $\mathfrak{M}$ is a group if I–VI hold. If I–V, VII hold, $\mathfrak{M}$ is an alternative system.

Interesting remarks are made concerning the geometric form of R as a subset of $\mathfrak{M}\times\mathfrak{M}\times\mathfrak{M}$. *J. L. Brenner.*

Grün, Otto. Beiträge zur Gruppentheorie. VI. Math. Nachr. **16** (1957), 271–280.

[For part V see Osaka Math. J. **5** (1953), 117–146; MR **15**, 852.] The author considers those groups which are

equal to their commutator subgroup. Let F be a free group. If σ is an isomorphism of F into F such that $F^\sigma(F, F)=F$ and (F^σ, F) is properly contained in (F, F), then $N=F^\sigma(F^\sigma, F) \neq F$ is a normal subgroup of F and $(F/N, F/N)=F/N$. Conversely let G be a group for which $(G, G)=G$, and represent G as a factor group F/N of a free group F. Then there exists an isomorphism σ of F into N such that $F^\sigma(F, F)=F$ and $(F^\sigma, F)\subset(F, F)$. (G, G) is the word subgroup generated by the word $X_1X_2X_1^{-1}X_2^{-1}$. The last section of the paper is concerned with groups that are equal to given word subgroups. The above results are elegantly generalized. *P. F. Conrad.*

Yacoub, K. R. Semi-special permutations. II. Semi-special permutations on $[p^\alpha]$. Duke Math. J. **24** (1957), 455–465.

The author determines all the semi-special permutations π on the group of integers, modulo p^α, where p is an odd prime and $\alpha > 1$.[For definitions and earlier results, see Yacoub, Proc. Glasgow Math. Assoc. **2** (1955), 116–123; **3** (1956), 18–35; MR **17**, 11; **19**, 5.] The cases $\alpha=2$, 3, 4 simplify considerably. For instance, if $\alpha=2$, 3, $\pi x = tx+p^{\alpha-1}vx(x-1)$, where t and v have to be chosen in a suitable fashion. *F. Haimo* (St. Louis, Mo.).

Blackburn, Norman. Über das Produkt von zwei zyklischen 2-Gruppen. Math. Z. **68** (1958), 422–427.

Let G be a group product of two cyclic 2-groups. Let G have descending central series $G\supseteq G_2\supseteq G_3\supseteq\cdots$, where $G_2=G'$, $G_i=[G_{i-1}, G]$, and let $\Phi(G)$ be the Frattini subgroup of G. If the commutator subgroup G' is not cyclic and if G/G' is of type $(n, 1)$, with $n>1$, then the author shows that $(G_2:G_3)=2=(G_3:\Phi(G'))$; that if the order of G' exceeds 4, then $(\Phi(G'):G_4)=2$; and that if $n>2$, G' is of type $(m, 1)$ for some suitable m. [See B. Huppert, Math. Z. **58** (1953), 243–264; MR **14**, 1059; and N. Itô, ibid. **62** (1955), 400–401; MR **17**, 125.] *F. Haimo.*

Itô, Noboru; et Ôhara, Akiko. Sur les groupes factorisables par deux 2-groupes cycliques. I. Cas où leur groupe des commutateurs est cyclique. Proc. Japan Acad. **32** (1956), 736–740.

Itô, Noboru; et Ôhara, Akiko. Sur les groupes factorisables par deux 2-groupes cycliques. II. Cas où leur groupe des commutateurs n'est pas cyclique. Proc. Japan Acad. **32** (1956), 741–743.

The authors determine the structure of a group G that can be factorized as a product of two cyclic p-groups with $p=2$. If the derived group G' of G is cyclic, and if G has a cyclic subgroup N that contains G' properly, then G also has a cyclic normal subgroup with a cyclic factor-group. If there is no such cyclic subgroup N, then G/G' is abelian of type $(2^\gamma, 2)$. If the derived group G' is not cyclic, then G/G' is also of type $(2^\gamma, 2)$. A large number of case distinctions are required. *K. A. Hirsch* (London).

Ikuta, Toshiharu. Über die Nichteinfachheit einer faktorisierbaren Gruppe. Nat. Sci. Rep. Lib. Arts Fac. Shizuoka Univ. no. **9** (1956), 1–2.

Let G be a finite group factorizable in the form $G=HP$, where P is cyclic of order p, p an odd prime. If the order of H is divisible by p, then H contains a proper normal subgroup of G. If not, then either the order of H is divisible by $p-1$ or P is normal in G. In particular, if G is of odd order, it cannot be simple. [For the case of a Fermat prime, this had been proved previously by Szép, Acta. Sci. Math. Szeged **14** (1952), 246; MR **14**, 722.] *K. A. Hirsch* (London).

Mitrinovitch, Dragoslav S. Sur certaines relations restant valables si l'on permute les opérateurs y intervenant. Bull. Soc. Math. Phys. Serbie **8** (1956), 15–22. (Serbo-Croatian summary)

The author proves special cases of the well-known "modular identity" in commutative lattice-ordered groups. *R. S. Pierce* (Seattle, Wash.).

Gravett, K. A. H. Ordered abelian groups. Quart. J. Math. Oxford Ser. (2) **7** (1956), 57–63.

The author gives a very simple proof of H. Hahn's well-known embedding theorem and completeness theorem on fully ordered abelian groups. The proof is based on the valuation theory of linear spaces discussed by the author in a former paper [same J. (2) **6** (1955), 309–314; MR **19**, 385]. [For recent proofs of Hahn's theorems see also P. F. Conrad, Amer. J. Math. **75** (1953), 1–29; MR **14**, 842; as well as M. Hausner and J. G. Wendel, Proc. Amer. Math. Soc. **3** (1952), 977–982; MR **14**, 566; and A. H. Clifford, ibid. **5** (1954), 860–863; MR **16**, 792.] *L. Fuchs* (Budapest).

Davis, Robert L. A special formula for the Lie character. Canad. J. Math. **10** (1958), 33–38.

Thrall [Amer. J. Math. **64** (1942), 371–388; MR **3**, 262] introduced the Lie character to study the structure of the free Lie ring. Brandt [Trans. Amer. Math. Soc. **56** (1944), 528–536; MR **6**, 146] showed that the Lie character can be expressed as $\zeta_n=n^{-1}\sum_{d|n}\mu(d)S_d^{n/d}$, where $\mu(d)$ is the Möbius function.

When the table of characters of the corresponding symmetric group is known, it is easy to express ζ in terms of S-functions, $\zeta=\sum C_\lambda\{\lambda\}$. To use this method for higher values of n would necessitate computing sizable parts of the corresponding table of characters.

The author obtains an explicit formula for C_λ for the case in which (λ) corresponds to a partition into exactly two parts. *D. E. Littlewood* (Bangor).

Croisot, R. Equivalences principales bilatères définies dans un demi-groupe. J. Math. Pures Appl. (9) **36** (1957), 373–417.

In analogy with the principal right and left equivalences of P. Dubreil [Mém. Acad. Sci. Inst. France **63** (1941), no. 3; MR **8**, 15] the author defines for any subset H of a semigroup (demigroupe) D the principal bilateral equivalence on D, $\mathscr{R}^1{}_H=\{(a, b): xay\in H\leftrightarrow xby\in H\}$. $\mathscr{R}^1{}_H$ is a congruence on D. Let $\mathscr{R}$ be any cancellative congruence on the semigroup D. Then $\mathscr{R}=\bigcap\{\mathscr{R}^1{}_H: H \text{ an } \mathscr{R}\text{-class}\}$. This paper gives a systematic study of the properties of principal bilateral equivalences and in particular investigates the relation between these congruences and the principal left and right equivalence of Dubreil [loc. cit.].

Write $H.a=\{(x, y): xay\in H\}$. H is a bilaterally strong subset of D if $(H.a)\cap(H.b)\neq\emptyset$ implies $H.a=H.b$. The bilateral residue of H is $W^1{}_H=\{x: H.x=\emptyset\}$. If non-empty, $W^1{}_H$ is a two-sided ideal of D. An ideal X of D is a bilateral residue of some set if and only if $X=(DXD\cdot D)\cdot D$ $(A\cdot B=\{x: BxC A\})$. H is bilaterally neat if $W^1{}_H=\emptyset$. If H is bilaterally strong and bilaterally neat then $\mathscr{R}^1{}_H$ is cancellative. D is bilaterally strict if every bilaterally strong subset is bilaterally neat. If H is a bilaterally strong subset of a bilaterally strict semigroup D then each $\mathscr{R}^1{}_H$-class A is bilaterally strong and $\mathscr{R}^1{}_H=\mathscr{R}^1{}_A$. If H is symmetric and strong with residue W and if $H\cap D^2\cap W=\emptyset$, then H is bilaterally strong and ${}_H\mathscr{R}=\mathscr{R}_H=\mathscr{R}^1{}_H$ [For definitions see Dubreil, loc. cit.] If S is a bilaterally strong

subsemigroup of D then the $\mathscr{R}^1_H$-class containing S is the unitary envelope [Dubreil, loc. cit.] U of S. If S is bilaterally neat then $D/\mathscr{R}^1_S$ is a group. If $W^1_S \neq s$ and U is symmetric then $D/\mathscr{R}^1_S$ is a group with zero. A characterization is obtained of those subsets H of D such that $D/\mathscr{R}^1_H$ is a group. A similar discussion is given of the more general case in which $D/\mathscr{R}^1_H$ is a cancellative semigroup containing a unique minimal two-sided ideal.

The paper contains many more results which expand the above outline. In particular there is a detailed discussion of the case when D is abelian and $D^2 \neq D$.

G. B. Preston (Shrivenham).

See also: Algebra: Miller.

Homological Algebra

Rees, D. Polar modules. Proc. Cambridge Philos. Soc. **53** (1957), 554–567.

Soit k un corps; on se donne une fois pour toutes un "anneau géométrique", i.e., une k-algèbre commutative A (à élément-unité) engendrée par un nombre fini d'éléments. On note d le degré de transcendance de A sur k; il est connu que si on écrit $A=S/\mathfrak{a}$, avec $S=k[X_1, \cdots, X_m]$, il existe des éléments $g_1, \cdots, g_{m-d} \in \mathfrak{a}$ tels que $(g_1, \cdots, g_{i-1}):g_i=(g_1, \cdots, g_{i-1})$ pour $1 \leq i \leq m-d$. On appelle "module géométrique" un A-module M de type fini; d'après le th. 2.1 de D. Rees [mêmes Proc. **52** (1956), 605–610; MR **18**, 277], on a (1) $\mathrm{Ext}_S^i(M, S)=0$ pour $i<m-d$, $\mathrm{Ext}_S^i(M, S) \approx \mathrm{Ext}_B^{i-m+d}(M, B)$ pour $i \geq m-d$, où $B=S/\mathfrak{g}$, $\mathfrak{g}$ étant l'idéal engendré par les g_i. Le même théorème montre que, pour tout entier r, $\mathrm{Ext}_S^{m-r}(M, S)$ est indépendant du choix de S; c'est un foncteur contravariant $T^r(M)$ du A-module M, nul pour $r<0$, et aussi pour $r>d$ d'après (1).

On étudie surtout $T^d(M)=M^* \approx \mathrm{Hom}_B(M, B)=\mathrm{Hom}_A(M, \mathrm{Hom}_B(A, B)) \approx \mathrm{Hom}_A(M, A^*)$, qu'on appelle le "module polaire" de M. Le noyau de l'homomorphisme naturel $M \to M^{**}$ est l'intersection de celles des composantes primaires de 0 dans M dont les idéaux premiers associés $\mathfrak{p}_i$ ont la dimension d (i.e.: $A/\mathfrak{p}_i$ a le degré de transcendance d); de plus $M^* \to M^{***}$ est un isomorphisme; en particulier, pour que $M \to M^{**}$ soit un isomorphisme, il faut et il suffit que M soit "équidimensionnel". Supposons que A soit équidimensionnel, et que les composantes primaires de l'idéal 0 soient irréductibles; alors pour tout idéal $\mathfrak{b}$ de A contenant des non-diviseurs de 0, il existe un idéal $\mathfrak{b}_1$ de A isomorphe (comme A-module) à $\mathfrak{b}^*$, et un tel idéal est "unmixed" ("ungemischt") de rang 1 ou est A. L'auteur donne d'autres résultats, notamment au sujet des invariants de Loewy des modules M et M^*. Il indique enfin comment les théorèmes ci-dessus doivent être précisés dans le cas de modules géométriques gr-a dués sur un anneau géométrique gradué A.

H. Cartan (Paris).

THEORY OF NUMBERS

General Theory of Numbers

Long, Calvin T. Note on normal numbers. Pacific J. Math. **7** (1957), 1163–1165.

It is proved that a number α is normal to base r if and only if there is an infinite set M of positive integers m_k such that α is simply normal to each of the bases r^{m_k}. The condition does not suffice if M is finite.

H. S. Zuckerman (Seattle, Wash.).

Paasche, Ivan. Über eine invariante Eigenschaft eines alternierenden Summationsprozesses bei wachsender Abschnittslänge der Ausgangsfolge. Math. Ann. **134** (1957), 95–100.

This paper deals with a variation of the procedure used in the theorem of Moessner [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 29; MR **14**, 353]; the variation consists in alternating the signs at each of the k steps, before taking the sums. Starting with the sequence 1, 2, 3, $\cdots$ (as in Moessner's theorem), the final sequence is proved to be 1, ± 2, 3, ± 4, $\cdots$ where the + signs hold when $k \equiv 0, 1 \pmod 4$ and the − signs hold when $k \equiv 2, 3 \pmod 4$. The method of proof is the same as that used by the present author [Arch. Math. **6** (1955), 194–199; MR **17**, 237] in his proof of Moessner's theorem.

H. W. Brinkmann (Swarthmore, Pa.).

Roberts, J. B. Note on linear forms. Proc. Amer. Math. Soc. **7** (1956), 465–469.

Frobenius proposed the problem of finding bounds $G(a_0, a_1, \cdots, a_s)$ such that the Diophantine equation $a_0x_0+a_1x_1+\cdots+a_sx_s=N$ has solutions in non-negative integers $x_0, x_1, \cdots, x_s$ for every $N \geq G(a_0, a_1, \cdots, a_s)$. Such bounds were obtained by A. Brauer [Amer. J. Math. **64** (1943), 299–312; MR **3**, 270] and by A. Brauer and B. Seelbinder [ibid. **76** (1954), 343–346; MR **15**, 777]. In particular, in the first of these papers, the best possible bound was found for the special case that the a_ν are consecutive integers. The author generalizes this result for the case that the a_ν are in arithmetic progression. He proves that every integer $N_0 \geq N=\{[a_0-2)/s]+1\}a_0+(d-1)(a_0-1)$ can be represented by $F=a_0x_0+(a_0+d)x_1+\cdots+(a_0+sd)x_s$ with non-negative x_ν if $(a_0, d)=1$, while $N-1$ cannot be represented. {The proof of the author can be simplified considerably. Let g be the greatest integer for which (*) $(a_0+sd)(g-d)<ga_0-d=K$. Then $K \equiv ga_0 \pmod d$. Assume that K could be represented by F with $x_0+x_1+\cdots+x_s=m$. Then $K \equiv ma_0 \pmod d$.] Hence $m \equiv g \pmod d$ since $(a_0, d)=1$. Since $(a_0+sd)(g-d)$ is the greatest number represented by F with $m=g-d$, and ga_0 is the smallest number represented with $m=g$, it follows from (*) that K is not represented, while all numbers greater than K and congruent to $K \pmod d$ are represented by F. From (*), we obtain $g<(a_0-1)/s+d$; hence $g=[(a_0-2)/s]+d$ and $K=a_0[(a_0-2)/s]+d(a_0-1)$. Now, let L be an integer greater than K. We determine h such that $L \equiv ha_0 \pmod d$ and $(h-d)a_0 \leq L < ha_0$; hence $L \leq ha_0-d$. If $h>g$, then $(a_0+sd)(h-d) \geq ha_0-d$, and all numbers congruent to L and greater than K are represented by F. Assume now that $h<g$. Then $L \leq ha_0-d < ga_0-d=K$. This is impossible. Therefore all numbers greater than K can be represented by F.} *A. Brauer.*

Roberts, J. B. On a Diophantine problem. Canad. J. Math. **9** (1957), 219–222.

It is proved that the Diophantine equation $mx_1+(m+a)x_2+(m+b)x_3=N$ always has solutions in non-negative integers x_1, x_2, x_3 if

$$N \geq m(b-2+[m/b])+(a-1)(b-1).$$

This bound for N is in some cases less and in others greater than the bound obtained by A. Brauer [Amer. J. Math. 64 (1942), 299–312; MR 3, 270; 5, 328; see also the preceding review]. *A. Brauer.*

Mordell, L. J. On the number of solutions in incomplete residue sets of quadratic congruences. Arch. Math. 8 (1957), 153–157.

Let p be an odd prime, and let $f(x, y)$ be an inhomogeneous quadratic form with integer coefficients, whose discriminant is not divisible by p. Let $h, k, \alpha, \beta, \gamma, \delta, \mu, \nu$ be integers, $0<h\leqq p$, $0<k\leqq p$, $\alpha\delta-\beta\gamma\not\equiv 0 \pmod p$. Let N denote the number of solutions of

$$f(\alpha\xi+\beta\eta+\mu, \gamma\xi+\delta\eta+\nu)\equiv 0 \pmod p,\ 0\leqq\xi<h,\ 0\leqq\eta<k.$$

The author shows that

$$|N-hk/p|<1+4p^{\frac{1}{2}}\log p+2p^{\frac{1}{2}}(\log p)^2$$

if f is equivalent to a canonical form ax^2+by^2+c ($abc\not\equiv 0 \pmod p$), and $|N-hk/p|<2p^{\frac{1}{2}}\log p+p^{\frac{1}{2}}(\log p)^2$ in the remaining case (equivalence with the canonical form y^2-x). In the first case the author uses Weil's inequality for Kloosterman sums. *N. G. de Bruijn.*

Moser, L.; and Wyman, M. Stirling numbers of the second kind. Duke Math. J. 25 (1957), 29–43.

Il existe une vaste littérature relative aux nombres $\sigma_n^m=[(-1)^m/m!]\sum_{k=1}^{m}(-1)^k\binom{m}{k}k^n$, connus sous le nom de nombres de Stirling II. Plusieurs expressions asymptotiques pour le calcul de ces nombres, quand n est relativement grand, ont été proposées. Certaines donnent une bonne approximation seulement si m reste petit par rapport à n. Ainsi les formules de Charles Jordan [Calculus of finite differences, 2nd éd., Chelsea, New York, 1950, pp. 173–174] donnent des valeurs inacceptables pour $n=50$, $m=15$. D'autres ne sont applicables que pour des valeurs fixées de m. Ces formules sont ici généralisées et perfectionnées. Un premier résultat est $\sigma_{n+k}^n=$

$$\binom{n+k}{k}(\tfrac{1}{2}n)^k\Big[1+\frac{(k)_2}{6n}+\frac{(k)_4}{72n^2}+\Big(\frac{(k)_6}{1296}-\frac{(k)_4}{180}\Big)\frac{1}{n^3}+\cdots\Big];$$

en prenant quatre termes, il donne σ_{50}^{10} avec une approximation suffisante, mais conduit à des calculs impraticables s'il faut tenir compte de cinq termes et plus. Une formule plus exacte,

$$\frac{n!(\exp(R)-1)^m}{2R^nm!(\pi mRH)^{\frac{1}{2}}}\Big[1-\frac{1}{mR}\Big\{\frac{15(C_3(R))^2}{16R^2H^3}-\frac{3C_4(R)}{4RH^2}\Big\}\Big],$$

est alors obtenue, dans laquelle R est solution de $R(1-e^{-R})^{-1}=n/m$, et H solution de $2(e^R-1)^2H=e^R(e^R-1-R)$, et où C_3 et C_4 sont des polynomes en R et en S, avec $S=(e^R-1)^{-1}$. Pour $m=35$, $n=50$, la formule donne 2.42357×10^{31} au lieu de 2.42355×10^{31}. *A. Sade.*

Swetharanyam, S. On sums involving the fractional parts of numbers. Proc. Indian Acad. Sci. Sect. A. 45 (1957), 385–389.

Let $\{x\}$ denote the fractional part of x. The author proves that

$$\text{(1)}\qquad \sum_{n\leqq x}\Big\{\frac{x}{n}\Big\}=(1-\gamma)x+O(x^{\frac{1}{2}}),$$

$$\text{(2)}\qquad \sum_{n\leqq x}n\Big\{\frac{x}{n}\Big\}=\Big(1-\frac{\pi^2}{12}\Big)x^2+O(\log x),$$

$$\text{(3)}\qquad \sum_{n\leqq x}n^k\Big\{\frac{x}{n}\Big\}=\Big(\frac{1}{k}-\frac{\zeta(k+1)}{k+1}\Big)x^{k+1}+O(x^k),\ k\geqq 2.$$

Results (1) and (2) are easily deduced from the well-known estimates

$$\sum_{n\leqq x}d(n)=x\log x+(2\gamma-1)x+O(x^{\frac{1}{2}}),$$

$$\sum_{n\leqq x}\sigma(n)=\frac{\pi^2x^2}{12}+O(x\log x),$$

while (3) follows from

$$\sum_{n\leqq x}n^k\Big\{\frac{x}{n}\Big\}=x\sum_{n\leqq x}n^{k-1}-\sum_{n\leqq x}\sigma_k(n),$$

the estimate

$$\sum_{n\leqq x}\sigma_k(n)=\frac{\zeta(k+1)}{k+1}x^{k+1}+O(x^k),\ k\geqq 2,$$

and a simple application of the Euler-Maclaurin sum formula. *M. Newman* (Washington, D.C.).

Erdös, P. On the growth of the cyclotomic polynomial in the interval (0, 1). Proc. Glasgow Math. Assoc. 3 (1957), 102–104.

Let $F_n(x)$ be the irreducible polynomial with the root $e^{2\pi i/n}$. The author gives a simple proof of the existence of a constant c such that $\max_{0\leqq x\leqq 1}|F_n(x)|>\exp\{n^{c/\log\log n}\}$ for infinitely many n. This is proved for n a product of the consecutive primes in an interval $(t, t+t^{1/4})$ by considering separately those factors $(x^d-1)^{\mu(n/d)}$ of $F_n(x)$ for which the number of divisors of d exceeds or is less than half the number of prime factors of n. It is conjectured that the above result holds for every $c<\log 2$. This would imply that an inequality of Bateman [Bull. Amer. Math. Soc. 55 (1949), 1180–1181; MR 11, 329] is the best possible. *D. H. Lehmer* (Berkeley, Calif.).

See also: Computing Machines: Balasiński and Mrówka.

Analytic Theory of Numbers

Turán, Paul. Über eine neue Methode der Analysis. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 5 (1955/56), 275–280. (Russian, English and French summaries)

In this expository lecture the author formulates four fundamental extremal problems which are in some respects more general than those of his book [Eine neue Methode in der Analysis und deren Anwendungen, Akad. Kiadó, Budapest, 1953; MR 15, 688]. Some refinements found after publication of this book are announced without proof. Part of this material will be published in a forthcoming joint paper with Vera T. Sós. *N. G. de Bruijn* (Amsterdam).

Turán, Paul. Über eine Anwendung einer neuen Methode auf die Theorie der Riemannschen Zetafunktion. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 5 (1955/56), 281–285. (Russian, English and French summaries)

In this lecture the author derives the "density hypothesis" from assumptions which are essentially weaker than Lindelöf's hypothesis. A more complete account of almost the same result was given in a later lecture, a report of which has appeared [J. Indian Math. Soc. (N.S.) 20 (1956), 17–36; MR 18, 792]. *N. G. de Bruijn.*

Zulauf, Achim. On sums and differences of primes and squares. Compositio Math. 13 (1958), 103–112.

The paper deals with the representations of an integer

q as a sum in which each term is either a prime in a given class of residues, or minus such a prime, or a given (positive or negative) integer times a square. The author restricts himself here to the case in which there is at least one positive and one negative term, and the number of prime terms plus half the number of square terms is greater than 2. He obtains an asymptotic formula for the number $N(q, n)$ of such representations in which either the sum of the positive terms or the sum of the negative terms is numerically less than or equal to n, and $n\to\infty$. *T. Estermann* (London).

Bateman, Paul T. Proof of a conjecture of Grosswald. Duke Math. J. **25** (1957), 67–72.

For $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ let $F(n)=\alpha_1+\cdots+\alpha_r$ and $a(n)=2^{F(n)}$. Grosswald [same J. **23** (1956), 41–44; MR **17**, 588] proved with certain absolute constants A, B as $x\to\infty$

$$\sum_{n\le x}{}' a(n)=Ax\log x+Bx+O(x^c) \tag{1}$$

with a certain $c<0.84$ and where Σ' denotes summation over the odd positive integers n. Furthermore he conjectured that (1) holds with the exponent $\log 2/\log 3$. The author proves this conjecture, which gives a best possible result. For the analogous problem with summation over all positive integers an error in the paper of Grosswald is corrected. By the same method further results on expansions of $\sum_{n\le x}^{(k)} a(n)$ (summation over the positive integers n not divisible by any of the first k primes) are obtained. *H.-E. Richert* (Göttingen).

See also: General Theory of Numbers: Swetharanyam. **Theory of Algebraic Numbers:** Schulz-Arenstorff. **Trigonometric Series and Integrals:** Erdös and Taylor.

Theory of Algebraic Numbers

Schulz-Arenstorff, Richard. Über die zweidimensionale Verteilung der Primzahlen reell-quadratischer Zahlkörper in Restklassen. J. Reine Angew. Math. **198** (1957), 204–220.

A well-known theorem of Walfisz [Math. Z. **40** (1935), 592–607] states that if $(a, r)=1$, and if $\pi(x; a, r)$ denotes the number of primes p with $p\le x$, $p\equiv r \pmod a$, then

$$\pi(x; a, r)=(\varphi(a))^{-1}\int_2^x (\log u)^{-1}du+O(x\exp(-c(\log x)^{\frac{1}{2}}))$$

uniformly for $x>3$, $a\le(\log x)^l$, where l is an arbitrary constant. The author generalizes this to a real quadratic number field K (with class number h and fundamental unit ε_0). An integer $\omega\in K$ is called a prime if the principal ideal (ω) is a prime ideal. ω' denotes the conjugate of ω. Let x_1 and x_2 be positive numbers, $x_1x_2=X$. Let $\mathfrak{a}$ be an integral ideal, and let ρ be an integer of K, with $(\rho, \mathfrak{a})=1$. $\pi(x_1, x_2; \mathfrak{a}, \rho)$ denotes the number of primes ω with $0<\omega<x_1$, $0<\omega'<x_2$, $\omega\equiv\rho \pmod{\mathfrak{a}}$. Let l be a fixed positive number. Then it is proved that

$$\pi(x_1, x_2; \mathfrak{a}, \rho)=(2h\varphi(\mathfrak{a})\log\varepsilon_0)^{-1}\log X\int_2^X (\log u)^{-2}du$$
$$+O(X\exp(-c(\log X)^{\frac{1}{2}})),$$

for $X=x_1x_2\ge 3$, $N\mathfrak{a}\le(\log X)^l$. The number c and the constant implied in the O-symbol only depend on l and on the field K.

For a fixed ideal $\mathfrak{a}$ this formula is due to Rademacher [Acta Arith. **1** (1935), 67–77; Trans. Amer. Math. Soc. **39** (1936), 380–398]. One of the main tools in Walfisz's case consists of Siegel's estimate $L(1, \chi)>a^{-\varepsilon}$ for the real characters. In the present case the author proves a similar inequality by application of R. Brauer's formula [Amer. J. Math. **69** (1947), 243–250; MR **8**, 567] $\log r(k)=o(\log|D|)$ ($r(k)$=residue of the Dedekind zeta function of the field k at $s=1$, D=discriminant of k; k runs through a sequence of fields with fixed degree, such that $|D|\to\infty$) to certain class fields over K, corresponding to $\mathfrak{a}$ and χ. *N. G. de Bruijn* (Amsterdam).

Lewis, D. J. Strongly definite polynomials. Michigan Math. J. **4** (1957), 187–191.

A polynomial $F(X)=F(x_1, x_2, \cdots, x_n)$ over a ring R is said to be a definite polynomial over R provided it has the property that its only zero in R is the trivial one. Chevalley [Abh. Math. Sem. Hamburg. Univ. **11** (1935), 73–75] proved that for definite polynomials over finite fields, the number of indeterminates cannot exceed the degree of the polynomial. Let K be a field which is complete under a discrete valuation, and which has a finite residue class field. Let $\mathfrak{o}$ be the ring of integers of K, $\mathfrak{p}$ the prime ideal in $\mathfrak{o}$, π a prime in $\mathfrak{p}$, and q the number of elements in the residue class field $\mathfrak{o}/\mathfrak{p}$. A polynomial $F(X)$ of degree d over $\mathfrak{o}$ is said to be strongly definite over $\mathfrak{o}$ provided that, for a_i in $\mathfrak{o}$, $F(a_1, a_2, \cdots, a_n)\equiv 0 \pmod{\mathfrak{p}^d}$ if and only if $a_i\equiv 0 \pmod{\mathfrak{p}}$ ($i=1, 2, \cdots, n$). Employing the method of Chevalley the author proves the following theorem. There exist polynomials $\phi_d(y)$, of degree $d-1$ over the ring of rational integers, such that if $F(X)$ is a strongly definite polynomial over $\mathfrak{o}$, of degree $d(d<q)$, then $n\le d^2\phi_d(q-1)$. *A. L. Whiteman* (Los Angeles, Calif.).

Masuda, Katsuhiko. Certain subgroup of the idèle group. Proc. Japan Acad. **33** (1957), 70–72.

Let k be an algebraic number field of finite degree, I the group of idèles of k and C the idèle class group of k. The author proves the following theorem. Let J be a subgroup of I such that each idèle $\mathfrak{a}=(\alpha_\mathfrak{g})$ in J has components $\alpha_\mathfrak{g}=1$ except for $\mathfrak{g}$ in a null set (with reference to Kronecker density). Then the image of J into C by the natural homomorphism has no element common with the connected component D of C other than 1. *Y. Kawada* (Tokyo).

Geometry of Numbers

Stöhr, Alfred. Über die Anzahl der Wege in einem gewissen Strassennetz. Math. Z. **68** (1957), 77–81.

Consider a street net composed of a system of equally spaced North-South and East-West directed streets. Let an intersection denote the origin P_{00} and the lattice point of the street intersections with integral coordinates (r, s) be denoted by P_{rs}. The paper deals with the number of ways one can proceed from P_{00} to P_{rs} in a direct path.

A route (Sektorweg) is a direct path which lies below the diagonal joining P_{00} to P_{rs}, which may include some lattice points on the diagonal. A primitive route is a route which does not contain any lattice point on the diagonal. By means of generating functions

$$f(z)=\sum_{n=1}^{\infty}\frac{1}{nr_0+ns_0}\binom{nr_0+ns_0}{nr_0}z^n,$$

$$g(z)=\sum_{n=0}^{\infty} b(nr_0, ns_0)z^n,$$

$$h(z)=\sum_{n=1}^{\infty} c(nr_0, ns_0)z^n,$$

where $b(r, s)$ and $c(r, s)$ denote respectively the total number of routes and primitive routes from P_{00} to P_{rs}, and r_0, s_0 two fixed and relatively prime positive integers, the author proves: $e^{f(z)}=g(z)=1/(1-h(z))$ and in particular $b(n, ns_0)=(1+ns_0)-1\binom{n+ns_0}{n}$. *Shen Lin.*

Schmidt, Wolfgang. On the convergence of mean values over lattices. Canad. J. Math. **10** (1958), 103–110.

The reviewer [Acta Math. **94** (1955), 249–287; MR **17**, 715] recently obtained a very complicated (but nevertheless useful) formula for certain mean values taken over the space of all lattices of determinant 1, but he had to adopt the convention that both sides might take the value $+\infty$. The author now shows that under appropriate conditions, both sides are finite; he also shows how the infinite sums involved can be rearranged in a more convenient form.

C. A. Rogers (Birmingham).

ANALYSIS

See: Analytic Theory of Numbers: Turán.

Functions of Real Variables

Baiada, Emilio. La variazione totale, la lunghezza d'una curva e l'integrale del calcolo delle variazioni in una variabile. Atti Accad. Naz. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 584–588.

This paper contains an announcement of the results proved by Baiada and Cardamone [Ann. Scuola Norm. Sup. Pisa (3) **11** (1957), 29–71; MR **19**, 945], and also a related definition for a generalized integral along a parametric curve $x=x(t)$, $y=y(t)$, namely,

$$\lim_{h\to 0}\int_{t_0}^{t_1} f\Big[x(t), y(t), \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}\Big]\,dt.$$

It is stated that this definition is invariant under change of parametric representation, gives the same value as the Weierstrassian definition, and reduces to the Lebesgue integral $\int_{t_0}^{t_1} f[x(t), y(t), x'(t), y'(t)]dt$ when the representation is absolutely continuous. The closing sentence is a remark on analogous considerations for non-parametric integrals.

L. M. Graves (Chicago, Ill.).

Redheffer, Raymond M. The maximum modulus. Monatsh. Math. **62** (1958), 76–83.

If the real function $u(x)$ is given on the unbounded connected set R of the euclidean n-space, then for every family S_r of bounded sets with boundaries B_r and such that $S_r \subset S_{r'}$ for $r<r'$ and $R\lim_{r\to\infty} S_r$, the "maximum modulus" of $u(x)$ is defined by $m(r)=\sup u(x)$, for $x\in R\cap B_r$. The behavior of the function $m(r)$ for $r\to\infty$ is studied under the hypothesis that $u(x)$ satisfies a "weak maximum principle", i.e., for every compact set $S\subset R$ and any constant k, the inequality $u(x)\leq k$ on the boundary of S implies $u(x)\leq k$ throughout S. Some of the properties obtained are independent of the election of the S_r and some are not. For example, under additional restrictions on $u(x)$, $m(r)$ is nondecreasing and $M=\lim_{r\to\infty} m(r)\leq\infty$ exists and is independent of the family S_r. Also, if $u(x)$ is continuous, conditions are established that guarantee the existence of polygonal arcs C extending to ∞ and such that $x\in C$ implies $M-\varepsilon\leq u(x)\leq M$ if $M<\infty$, or $u(x)>1/\varepsilon$ if $M=\infty$, for any given $\varepsilon>0$, or in some cases $\lim_{x\to\infty} u(x)=M$. *A. G. Azpeitia* (Providence, R.I.).

Dinghas, Alexander. Konvexitätseigenschaften von linearen und multilinearen Formen. Arch. Math. **8** (1957), 135–143.

The author shows how the classical inequalities of Hölder, Minkowski and others can be obtained from a consideration of the monotoneity and convexity properties of certain functionals $(\sum a_i x_i^r)^{1/r}$, where $\sum a_i=1$, $x_i>0$. The results are not new but the treatment is a unification and simplification of previous work. *H. G. Eggleston.*

Volpato, Mario. Sulla formula di Green nell'ambito delle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 161–167.

The author points out some extensions and corollaries of his Theorem 1, proved in Note I [same Rend. (8) **20** (1956), 30–37; MR **18**, 198]: the classical theorems on change of variables and integration by parts, an extension of the second mean value theorem (for the Lebesgue integral), and a sufficient condition for a function $F[\varphi(y), y]$, $c\leq y\leq d$, to be absolutely continuous and satisfy $dF[\varphi(y), y]/dy=F_x'[\varphi(y), y]\varphi'(y)+\int_{\varphi(c)}^{\varphi(y)} F_{xy}''(t, y)dt+F_y'[\varphi(c), y]$. The last condition is as follows: $F(x, y)$ is absolutely continuous in each variable in $R(a\leq x\leq b, c\leq y\leq d)$, F_x' is measurable in x and absolutely continuous in y for almost every x, F_{xy}'' is integrable in R, $|F_x'(x, y)|\leq M(x)$, where $M(x)$ is integrable in (a, b), and $F_x'[\varphi(y), y]\varphi'(y)$ is integrable in (c, d). *M. Cotlar.*

Volpato, Mario. Sulla formula di Green nell'ambito delle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 299–306.

The author gives another extension of his Theorem 1 [see the preceding review] and proves the formula

$$\int_{\alpha(y)}^{\beta(y)} f(t, y)dy=\int_{\alpha(c)}^{\beta(c)} f(t, c)dt+\int_c^y f[\beta(t), t]\beta'(t)dt-\int_c^y f[\alpha(t), t]\alpha'(t)dt+\int_c^y d\eta\int_{\alpha(\eta)}^{\beta(\eta)} f_y'(t, \eta)dt,$$

assuming the following conditions: (I) $\alpha(y)$ and $\beta(y)$ are absolutely continuous in $J(c\leq y\leq d)$ and satisfy $\alpha(y)<\beta(y)$; (II) $f(x, y)$ is defined in the set B: $\alpha(y)\leq x\leq\beta(y)$, $c\leq y\leq d$, is measurable in x, absolutely continuous in y, for almost all $x\in I(a\leq x\leq b)$, where $a=\min_J \alpha(y)$, $b=\max_J \beta(y)$, and $|f(x, y)|\leq M(x)$, where $M(x)$ is integrable in I; (III) the functions $f[\alpha(y), y]\alpha'(y)$ and $f[\beta(y), y]\beta'(y)$ are integrable, and $M(\alpha(y))\alpha'(y)$, $M(\beta(y))\beta'(y)$ are measurable in J; (IV) f_y' is integrable in B. *M. Cotlar.*

Tambs Lyche, R. A monotonously increasing continuous function which is not differentiable in any interval. Nordisk Mat. Tidskr. **5** (1957), 139–142, 168. (Norwegian. English summary)

A function with the properties of the title is given by

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}|x-\alpha_n| \quad (0\leq x\leq 1),$$

where $\{\alpha_n\}$ is the sequence of rational numbers between 0

and 1, arranged for instance by increasing denominators:

$$\alpha_1=0,\ \alpha_2=1;\ \alpha_3=\tfrac{1}{2};\ \alpha_4=\tfrac{1}{3},\ \alpha_5=\tfrac{2}{3};\ \alpha_6=\tfrac{1}{4},\ \alpha_7=\tfrac{3}{4};\ \cdots.$$

It is shown that $f(x)$ is differentiable only for irrational x, whereas for a rational value of the argument:

$$f'_+(\alpha_k)-f'_-(\alpha_k)=2^{-k+1}.$$

Author's summary.

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: Nevanlinna.

Measure, Integration

Morse, Marston; and Transue, William. Products of a C-measure and a locally integrable mapping. Canad. J. Math. **9** (1957), 475–486.

Let α be a Radon measure on a locally compact space E, y be a function on E which is locally α-integrable. A new Radon measure $\beta=y\cdot\alpha$ results from the definition $\int u d\beta=\int uy d\alpha$ $(u\in\mathscr{K}_+(E))$. This paper undertakes a detailed examination of relations between α- and β-integrability. The principal Theorem 1.1 states that if x is any function on E such that $\int^* |x||y|d\alpha$ and $\int^* |x|d\beta$ are both finite, then these upper integrals are equal; and if in addition, either of the integrals $\int xyd\alpha$ and $\int xd\beta$ exists, then so does the other, and with the same value. The application of this theorem, whilst free of difficulty when E is σ-finite, demands some care in the contrary case. The authors find a place for comments on this situation and give examples to show that if α and y are positive and E is not σ-finite, there exist functions $x\geqq 0$ such that either of $\int^* xyd\alpha$ and $\int^* xd\beta$ may be infinite whilst the other is finite. As the authors point out, there is some overlap with Bourbaki [Intégration, Ch. V, Actualités Sci. Ind., no. 1244, Hermann, Paris, 1956; MR **18**, 881], but it is none the less useful to have this brief independent account. *R. E. Edwards* (London).

Pełczyński, A.; and Rolewicz, S. Remarks on the existence of the Riemann-Stieltjes integral. Colloq. Math. **5** (1957), 74–77.

Let Φ be a real-valued function on the real line, and let $\langle a, b\rangle$ be a non-degenerate closed real interval. It is shown that if for every continuous real-valued function f on $\langle a, b\rangle$ the Riemann-Stieltjes integral $\int_a^b \Phi(f(x))df(x)$ exists, then $\Phi=$const. From this it follows immediately that there must be continuous functions f for which (*) $\int_a^b f(x)df(x)$ fails to exist, answering a question posed by C. Ryll-Nardzewski. Other results of the paper combine to show that among the real-valued functions f on $\langle a, b\rangle$ the ones for which the integral (*) exists are in fact precisely those whose square variation $\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))^2$ corresponding to a partition $a=x_0<x_1<\cdots<x_n=b$ of $\langle a, b\rangle$ tends to zero with then orm of the partition. *T. A. Botts* (Charlottesville, Va.).

Bogdanowicz, W. On singular integrals. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 247–249, XXI. (Russian summary)

This paper deals with singular integrals of the impulse or "Dirac Delta Function" type of the form $\int_a^b f_n(s,t)x(t)dt$, where $\{f_n(s,t)\}$ is a sequence of measurable functions of t in the interval $[a, b]$ for every fixed value of s in the same interval, and $x(t)$ is a function belonging to the class L^p $(p\geqq 1)$ over that interval, and where, for every point s in the interval $\lim_{n\to\infty}\int_a^b f_n(s,t)dt=1$, and $\lim_{n\to\infty}\int_c^d f_n(s,t)dt$, $=0$ for every closed interval $[c, d]\subset[a, b]$ that does not contain the point s.

The problem considered is that of finding necessary and sufficient conditions such that $\lim_{n\to\infty}\int_a^b f_n(s,t)x(t)dt=x(s)$ for every function $x(t)$ in L^p and for all points s in $[a, b]$ which are either the Lebesgue points of $x(t)$ (first case) or those points for which $\lim_{h\to 0} h^{-1}\int_s^{s+h} x(t)dt=x(s)$ (second case). The first case was given previously for the special case $p=1$ by Fadeev [Mat. Sb. N.S. **1(43)** (1936), 351–368]. The general results for this case involve spaces of essentially bounded functions and those for which the sum

$$\int_a^s \operatorname*{ess\,sup}_{v\in(a,s)}|x(v)|dt+\int_s^b \operatorname*{ess\,sup}_{v\in(t,b)}|x(v)|dt$$

is finite. The second case involves similar conditions concerning the space of functions of bounded variation.

Similar results are also obtained for sequences of integrals approaching the derivative of $x(t)$ at a fixed point.

The proofs will appear in Studia Mathematica and involve methods of functional analysis. *B. Lepson.*

Gordon, H.; and Lorch, E. R. The projection of a linear functional on the manifold of integrals. Canad. J. Math. **9** (1957), 465–474.

Let X be a set and B a linear space of bounded real-valued functions defined on X such that: 1) $\inf(f, g)\in B$ and $\sup(f, g)\in B$ if $f, g\in B$; 2) B is a Banach space for the norm $\|f\|=\sup_{x\in X}|f(x)|$. Let B' be the dual of B. An element $x'\in B'$ is a positive integral if for every $f\geqq 0$, $f\in B$ we have $x'(f)\geqq 0$ and if also for every increasing sequence $(f_n)_{1\leq n<\infty}$ of functions belonging to B, converging pointwise to $f\in B$, we have $\lim_{n\to\infty}x'(f_n)=x'(f)$. An element $x'\in B'$ is an integral if $x'=x_1'-x_2'$ where x_1' and x_2' are positive integrals. The main result of this paper is the following: There is a unique projection P of B' onto a subspace of B' such that: 1) $Px'=x'$ if and only if x' is an integral; 2) $Px'\geqq 0$ and $(I-P)x'\geqq 0$ if $x'\geqq 0$. A similar theorem is proved for complex spaces. These results may be considered as abstract formulations of the decomposition theorem of a finitely additive set function into a completely additive part and a purely finite part. [See also Heinz Bauer, J. Reine Angew. Math. **194** (1955), 141–179; MR **17**, 177.] *C. T. Ionescu Tulcea.*

Abian, Smbat. A general definition of convergence, continuity, differentiability and integrability. Boll. Un. Mat. Ital. (3) **12** (1957), 418–420.

Definitions are given in a very general setting of the notions of convergence, continuity, derivative, and integral. *J. Schwartz* (Upton, N.Y.).

Matthes, Klaus. Über eine Verallgemeinerung des Lebesgueschen Integralbegriffes. I. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **5** (1955/56), 287–295. (Russian, English and French summaries)

Let E be a locally compact Hausdorff space and $L_0(E)$ the family of all continuous real-valued functions on E which vanish outside compact sets. In this paper, the familiar derivation of a regular Borel measure on E from a positive linear functional on $L_0(E)$ is extended to the case that both the measure and the functional take values in a conditionally complete vector lattice V. (They are called a "Radon V-measure" and a "Radon V-functional", respectively.) This is illustrated by a few

examples, and is to be followed by the corresponding integration theory. *H. M. Schaerf* (Madison, Wis.).

See also: Functions of Real Variables: Baiada; Volpato. **General Topology:** Dekker and de Groot; Dekker. **Convex Domains, Integral Geometry:** Dinghas.

Functions of Complex Variables

Menger, Karl. The behavior of a complex function at infinity. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 512–513.

[Editor's note: This is a second review; the first appeared in MR **17**, 138.]

The author points out that the conventional device for discussing the behavior of an analytic function at ∞ fails to give an adequate picture of the structure of the function. He suggests the following improvement. Let f be a function whose domain is an open subset of the Riemann sphere, with values on the sphere. If a is an interior or boundary point of the domain of f, define

$$\mathbf{E}f(a)=\lim \frac{f(z_n')-f(z_n)}{z_n'-z_n}$$

if this limit exists (on the sphere) as $z_n\to a$ and $z_n'\to a$ through values not including either a or ∞, with $z_n'-z_n\to 0$. Higher derivatives are defined by recurrence. If $a\neq\infty$, and f is regular at a, then $\mathbf{E}f(a)$ coincides with the conventional derivative of f at a. However, the new definition makes the derivatives of functions at ∞ behave just as they do at finite points, so that for instance the derivative of the identity function is 1 at ∞ just as at finite points. If a is a point on the sphere where $\lim f$ exists and if neither (i) $\mathbf{E}^k f(a)=\infty$ for almost all $k>0$ nor (ii) $\mathbf{E}^k f(a)=0$ for almost all $k>0$ without $\mathbf{E}^k f$ being constant, then the author calls a an ordinary point for f. Using this terminology he can state Cauchy's theorem in the symmetric form: if f is ordinary in either complementary region of a rectifiable simple closed curve Γ and satisfies proper conditions on Γ, then $\int_\Gamma f=0$. He gives several other instances in which the new terminology leads to simplified statements in which it is no longer necessary to distinguish between finite points and ∞.

R. P. Boas, Jr. (Evanston, Ill.).

Macintyre, A. J. A direct proof of Morera's theorem. Arch. Math. **8** (1957), 374–375.

Let $\int f(z)dz=0$ over every closed curve in a simply connected domain D. Under this assumption, the author proves directly that $\int[f(\zeta)/(\zeta-z)]d\zeta$ vanishes if z lies outside the closed path of integration. This in turn leads in the usual way to the evaluation of the latter integral as $2\pi i f(z)$ if z is inside that path, and thence to the fact that $f(z)$ is analytic in D. *F. Herzog.*

Stallmann, Friedemann. Konforme Abbildung von Kreisbogenpolygonen. II, III. Math. Z. **68** (1957), 27–76; 245–266.

This series of papers deals with the conformal mappings of circular arc polygons onto half-planes which are determined by the quotients of two solutions of Fuchsian differential equations and those equations obtained from them through confluence. The exact solution of this old and difficult problem is not to be expected. By using asymptotic solutions of the differential equations, with special reference to the theory of Langer [Trans. Amer. Math. Soc. **37** (1935), 397–416], which are valid for large numerical values of the accessory parameters, the author determines approximate polygons which become more accurate as the parameters grow.

In paper I, the asymptotic theory was reviewed and applied to some special cases of the Lamé equation [Math. Z. **60** (1954), 187–212; MR **15**, 948]. In paper II, Gauss's and Whittaker's hypergeometric equations and the mappings engendered by their solutions are investigated. Gauss's equation has the self-adjoint form

$$\frac{y''}{y}=-\tfrac{1}{4}\Big[\frac{1-\delta_0^2}{x^2}+\frac{1-\delta_1^2}{(x-1)^2}-\frac{1+\delta_\infty^2-\delta_0^2-\delta_1^2}{x(x-1)}\Big].$$

The ratio of two of its solutions maps the upper half-plane onto the interior of a curvilinear triangle with angles $\pi\delta_0$, $\pi\delta_1$ and $\pi\delta_\infty$. The following configurations are treated: a) δ_0^2 and $\delta_1^2\leq\frac{1}{4}$; δ_∞ real or imaginary, and $\delta_\infty\to\infty$ or $\delta_\infty\to i\infty$, b) δ_1 and $\delta_\infty\to\infty$ or $i\infty$, with δ_1/δ_∞ real, c) all δ's become infinite, and are either real or pure imaginary.

The degenerate mappings given by solutions of Whittaker's equation and the Whittaker functions are discussed for (a) $m^2\leq 1/16$, k real or pure imaginary, $k\to\infty$ or $i\infty$; (b) k real, m real and m^2 less than, equal to, or greater than k^2; m imaginary; (c) k imaginary, m imaginary and $|m|^2<k^2$, $m^2=-k^2$, or $|m|^2>k^2$; m real. In paper III, solutions of the Lamé equation and the mappings they engender are investigated for various general configurations of the parameters. Numerous illustrations are given throughout. *N. D. Kazarinoff.*

Unkelbach, Helmut. Die funktionentheoretischen Invarianten für Ecken gerader Ordnung. Math. Ann. **133** (1957), 320–327.

The author [Math. Ann. **129** (1955), 391–414; **130** (1955), 327–336; MR **17**, 143, 956] has given a definition of angle at a vertex of a generalized circular polygon laid on a Riemann surface. For vertices of certain lower orders it has been shown that their magnitude can be expressed in terms of coefficients of a rational function contained in Schwarz' differential equation satisfied by the mapping function of the half-plane onto the polygon. In the present paper the author establishes a general formula of the same nature which is valid for a vertex of any even order. *Y. Komatu* (Tokyo).

Tsuji, Masatsugu. On a Riemann surface, which is conformally equivalent to a Riemann surface with a finite spherical area. Comment. Math. Univ. St. Paul **6** (1957), 1–7.

The author introduces the class O_{MD} of Riemann surfaces F with the following property. Every nonconstant meromorphic function $w(z)$ on F maps F onto a covering surface of the w-sphere with an infinite spherical area

$$I(w)=\iint_W\Big(\frac{|w'(z)|}{1+|w(z)|^2}\Big)^2dxdy.$$

The class differs basically from all others considered in the classification theory in that it contains no closed or open surfaces of finite genus.

The first question is whether or not there are any surfaces in O_{MD}. The author claims the affirmative, but an unfortunate error has crept into his proof. It is in the statement that there are boundary components which cannot be defined by a sequence of noncompact subregions whose relative boundaries consist of bounded

numbers of disjoint analytic Jordan curves. Actually, one single curve for each subregion will always suffice. For if there are several, they can be connected by the edges of suitably chosen narrow strips in the subregion. The author's proof thus collapses, but the good cause is not lost: Mizumoto [paper reviewed second below] has given an independent proof of the non-emptiness of O_{MD}.

The author also considers a hyperbolic covering surface F of the z-sphere and maps the universal covering surface of F onto $|\zeta|<1$. If $w(z)$ on F has a finite $I(w)$, then he shows that the Nevanlinna characteristic function of $v(\zeta)=w(z(\zeta))$ has the property $T(r, v)=O(1)$, $0\leqq r<1$. A related result is obtained for a harmonic function on F with a finite Dirichlet integral. *L. Sario.*

Tsuji, Masatsugu. On Abelian and Schottkyan covering surfaces of a closed Riemann surface. Comment. Math. Univ. St. Paul **6** (1957), 9–28.

Let $\varphi(z)$ be an Abelian integral of the first kind on a closed Riemann surface F. The unramified covering surface Φ of F which, in a natural sense, is the weakest covering surface rendering $\varphi(z)$ singlevalued, is called an Abelian covering surface of F. It is known that every such surface belongs to the class O_{AD}.

Since the new class O_{MD} of the author [see the paper reviewed above] is obviously contained in O_{AD}, the question arises as to whether or not the Abelian covering surfaces Φ are in O_{MD} as well. Let r be the greatest number of linearly independent periods of $\varphi(z)$ on F. The author's main statement is that $\Phi \in O_{MD}$ if $r\geqq 3$. However, the same deficiency as in the preceding paper invalidates his proof. The erroneous point here is the following conclusion by contradiction: "... there exists a decreasing sequence of non-compact subsurfaces of Φ, defining an ideal boundary point, whose compact boundary consists of at most N disjoint analytic Jordan curves, which is impossible, if $r\geqq 3$."

The cases $r\leqq 2$ are considered in detail, and a differential equation related to surfaces of Schottky type is discussed. *L. Sario* (Los Angeles, Calif.).

Mizumoto, Hisao. On Riemann surfaces with finite spherical area. Kōdai Math. Sem. Rep. **9** (1957), 87–96.

Given an open Riemann surface R, the author considers the following special covering surface W of R. Cut R along k disjoint nondividing cycles $\alpha_1, \alpha_2, \cdots, \alpha_k$ so as to form a surface F whose boundary consists of the (ideal) boundary of R and the edges α_i^+ and α_i^- $(i=1, \cdots, k)$ of the cycles α_i. Let F_n be an infinite collection of duplicates of F $(n=0, \pm 1, \pm 2, \cdots)$. Upon identifying α_i^+ of each F_n with α_i^- of F_{n+1}, one obtains a nonramified covering surface W of R. The author is interested in determining when W belongs to Tsuji's class O_{MD} [see the papers reviewed above]. By definition, every nonconstant meromorphic function $w(z)$ maps $W \in O_{MD}$ onto a covering surface of the w-sphere with a finite spherical area.

Consider an exhaustion $E=\{R_n\}$ of R. The boundary L_n of R_n is chosen to consist of a finite number of closed analytic Jordan curves, each dividing R into disjoint parts. Let $\lambda(L)_E$ be the extremal length of the family of sets of disjoint analytic Jordan curves homotopic to L_n. Denote by O'' the Kusunoki class of Riemann surfaces that possess an exhaustion with $\lambda(L)_E=0$ [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1956), 1–22; MR **19**, 846]. The author derives a sufficient modular condition for $R \in O''$ and $W \in O_G$. He then presupposes that R and W possess these properties and gives a necessary and sufficient condition for $W \in O_{MD}$. The condition is in terms of period matrices of R and forms a direct generalization of Ozawa's result for the case of a closed R [Kōdai Math. Sem. Rep. **8** (1956), 152–157; MR **19**, 259].

The role of O_{MD} in the general classification problem is then investigated. Let O_L be the Heins class of Riemann surfaces admitting no Lindelöfian meromorphic functions [Ann. of Math. (2) **55** (1952), 296–317; MR **13**, 643]. If the complements of O_G with respect to more comprehensive classes are indicated by primes, then it is shown that

$$O_{HB}' \subset O_L' \subset O_{MD}' \cap O_{AB}' \subset O_{MD}' \subset O_{MD}' \cup O_{AB}' \subset O_{AD}',$$

where all inclusions are strict. *L. Sario.*

Jurchescu, Martin. Recouvrements riemanniens définis par des équations différentielles du second ordre. C. R. Acad. Sci. Paris **245** (1957), 627–630.

Etude de la surface de Riemann $(R; f, \varphi)$ des integrales non algébriques $w(z)=f\circ\varphi^{-1}(z)$ d'une équation différentielle du type suivant:

$$w''=F_1(w, z)w'^3+F_2(w, z)w'^2+F_3(w, z)w'+F_4(w, z).$$

On ajoute l'hypothèse d'existence dans les plans (w) et (z) de deux domaines, respectivement G_w et G_z, ayant des frontières de capacité nulle, tels que les fonctions F_i soient méromorphes dans $G_w \times G_z$. Le recouvrement Riemannien $(R; f)$ du plan (w) possède alors la propriété d'Iversen et il est possible d'ajouter quelques précisions sur les éléments frontière de R. *L. Fourès* (Marseille).

Kaplan, Wilfred. Zeros of analytic functions and the moment problem. Ann. Acad. Sci. Fenn. Ser. A. I. no. **250/17** (1958), 11 pp.

Let $f(z)$ be an entire function, $f(0)\neq 0$. The author considers various connections between the expansion $-f'(z)/f(z)=\sum_{k=0}^{\infty}\sigma_k z^k$ (valid for $|z|$ sufficiently small) and the representation of $f'(z)/f(z)$ in terms of Stieltjes transforms (and also in terms of Laplace transforms). The following result is typical of those obtained by the author: $f(z)$ is of finite order, and has all its zeros on the real interval $[1, \infty)$, if and only if, for some N, $\{\sigma_k\}$ is completely monotonic for $k\geqq N$. *A. Edrei.*

Barry, P. D. The minimum modulus of certain integral functions. J. London Math. Soc. **33** (1958), 73–75.

Let $f(z)=\sum a_n z^n$ be an entire function, where a_n is ultimately non-zero and $|a_n/a_{n+1}|\geqq\theta^2|a_{n-1}/a_n|$ $(\theta>1)$ for all sufficiently large n. S. M. Shah proved [Bull. Amer. Math. Soc. **52** (1946), 1046–1052; MR **8**, 322] that if $\theta>2.2$, then

$$\limsup_{r\to\infty}\frac{m(r, f)}{M(r, f)}\geqq 1-C(\theta) \quad (0\leqq C(\theta)<1),$$

where $m(r, f)$ and $M(r, f)$ denote, respectively the minimum and maximum of $|f(z)|$ for $|z|=r$. The author obtains the asymptotic behavior, as $\theta\to\infty$, of the best possible value of $C(\theta)$. *A. Edrei* (Syracuse, N.Y.).

Constantinescu, Corneliu. Über die defekten Werte der meromorphen Funktionen deren charakteristische Funktion sehr langsam wächst. Compositio Math. **13** (1958), 129–147.

The author shows that, if a meromorphic function belongs to certain special classes of functions of lower order zero, then it cannot have more than one deficient value.

The restriction to these classes is unnecessary, since it is

known that the property holds for all meromorphic functions of lower order zero [A. Edrei and W. H. J. Fuchs, Bull. Amer. Math. Soc. **63** (1957), 366]. *A. Edrei.*

König, Heinz. Über die Landausche Verschärfung des Schottkyschen Satzes. Arch. Math. **8** (1957), 112–114.

In the seventh chapter of his book, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie [2nd ed., Springer, Berlin, 1929], Edmund Landau has shown how his extension of Schottky's theorem [loc. cit., Theorem 6] can be derived from Bloch's theorem [loc. cit., Theorem 1]. In order to obtain the identical result, the author of the present paper follows Landau's procedure in all essentials, with one minor change: For the principle of "condensation of exceptional values" [loc. cit., Theorem 2], he uses the fact that, if $\phi(z)$ omits the values ± 1, then $(1/\pi)\arccos[(1/\pi)\arccos\phi(z)]$ omits a sufficiently "dense" lattice of points. *F. Herzog.*

Erdös, Paul. Über eine Fragestellung von Gaier und Meyer-König. Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 89–92.

For a function $f(z)=\sum a_n z^n$, regular and unbounded in the unit disk D, a point $e^{i\theta}$ is B-singular provided f is unbounded in each open sector of D which contains the radius of $e^{i\theta}$. It was shown by Gaier and Meyer-König that several classical theorems on singularities of power series can be transferred to B-singularities [same Jber. **59** (1956), Abt. 1, 36–48; MR **18**, 385]. The author shows that the Fabry gap theorem can not be transferred. His counter-example is of the form

$$f(z)=\sum z^{m_k p_k} f_k(z)/(m_k+1)k;$$

here $f_k(z)=1+z^{p_k}+z^{2p_k}+\cdots+z^{m_k p_k}$, p_k denotes the kth prime, and $\{m_k\}$ is any sequence of natural numbers which grows sufficiently fast. *G. Piranian.*

Gabriel, R. F. A generalized Schwarzian derivative and convex functions. Duke Math. J. **24** (1957), 617–626.

Continuing earlier work [Proc. Amer. Math. Soc. **6** (1955), 58–66; MR **16**, 807] the generalised Schwarzian derivative is defined as

$$\{f, z, n\}=\left(\frac{f''(z)}{f'(z)}\right)'-\frac{1}{n+1}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

After some discussion of the properties of this Schwarzian derivative it becomes necessary to establish the inequality

$$r\int_0^r[\gamma'(\rho)]^2 d\rho - cr\int_0^r[\gamma(\rho)]^2 d\rho/\rho \geqq rg(r)[\gamma(r)]^2$$

for a certain class of functions $\gamma(\rho)$. The function $g(r)$ is related to Bessel functions and equality is attained by a $\gamma(\rho)$ also related to Bessel functions. A sequence of constants c_n is defined in terms of the smallest positive root of an equation involving Bessel functions. If now $f(z)=z^{-n}+\cdots$ is analytic and single-valued and $f'(z)\neq 0$ in $0<|z|<1$ and $|\{f, z, n\}|\leqq(n+1)c_n/|z|$, then the map of $|z|=r<1$ is convex of order $-n$. If further $f(z)\neq w_0$, then $f(z)-w_0$ is n-valent and starlike. *A. J. Macintyre.*

Ozawa, Mitsuru. A distortion theorem on schlicht functions. Kōdai Math. Sem. Rep. **9** (1957), 145–157.

Using Schiffer's variational methods, the author obtains an upper bound for $|f'(r)+f'(-r)|/|f(r)-f(-r)|$ for all sufficiently small r. This result is easily shown to be equivalent to Löwner's inequality $|a_3|\leqq 3$. The author states in a note added in the proof that he was unaware that the problem had been completely solved for all r by G. M. Golusin [Mat. Sb. N.S. **18(60)** (1946), 379–390; MR **8**, 574]. *G. Springer* (Lawrence, Kans.).

Oikawa, Kôtaro. A distortion theorem on schlicht functions. Kōdai Math. Sem. Rep. **9** (1957), 140–144.

The author obtains precise upper and lower bounds for the spherical derivative $Df(z)=|f'(z)|/(1+|f(z)|^2)$ for the class of univalent functions which are regular in the unit circle and normalized so that $f(0)=0$ and $f'(0)=1$. He uses Schiffer's variational methods to obtain these results. *G. Springer* (Lawrence, Kans.).

Tietz, H. Faber series and the Laurent decomposition. Michigan Math. J. **4** (1957), 175–179.

Let C be an analytic Jordan curve in the z-plane. Let $t=f(z)$ map the exterior region $A(C)$ of C conformally onto $|t|>k$ with $f(\infty)=\infty$. Let F be the limit of a Faber series $F(z)=\sum_0^\infty a_n F_n(z)$ which converges uniformly in any compact subset of the interior region of C. Associated with F is a Laurent series in f of the form $F(z)=\sum_{-\infty}^{\infty} b_n[f(z)]^n$, where $b_n=a_n$ if $n\geqq 0$. The author's main theorem states that if $g(z)=\sum_0^\infty a_n[f(z)]^n$, then $F(z)-g(z)$ can be continued analytically throughout $A(C)\cup C$, where it is represented by the series $\sum_0^\infty a_n[F_n(z)-f(z)^n]$, and it vanishes at ∞. This is a generalization of a result of J. Ullman [same J. **2** (1954), 109–114; MR **16**, 121]. The series $\sum_0^\infty a_n F_n$ and $\sum_0^\infty a_n/n$ have the same convergence behavior in $A(C)\cup C$. These methods are applied to give a proof of a theorem of P. Heuser [Math. Z. **38** (1934), 777–782]. *G. Springer* (Lawrence, Kans.).

Fekete, M.; and Walsh, J. L. Asymptotic behavior of restricted extremal polynomials and of their zeros. Pacific J. Math. **7** (1957), 1037–1064.

This is an extensive study of polynomials which have been restricted by the specification of their first k coefficients (k not necessarily constant) and which are extremal with respect to general norm N. For a given compact set S, let $\tau(S)$ be its transfinite diameter and $G(z)$ be the Green's function with pole at infinity of the maximal infinite region K containing no point of S. Let $A_n=A_n(\gamma_j, 1\leqq j\leqq k)$ denote the set of all polynomials $z^n+a_{1n}z^{n-1}+\cdots+a_{nn}$ satisfying $a_{jn}=\gamma_j$, $1\leqq j\leqq k$, $1\leqq k\leqq n-1$. The authors deal with continuous quasi-Tchebycheff (q.T.) norms N over S [for q.T. norms see Fekete and Walsh, J. Analyse Math. **4** (1955), 49–87, p. 53; MR **17**, 354], a concept which generalizes classical norms such as $\max_{z\in S}|f(z)|$ and various integral norms. For each $\gamma_j=\gamma_j(n)$ and $k=k(n)$, there is a polynomial $A_n(z, N)$ of least N-norm in $A_n(\gamma_j, 1\leqq j\leqq k)$. Set $\nu_n=N[A_n(z, N), S]$ and $|\zeta(z)|=e^{G(z)}\tau(S)$. In part I, conditions are given for the validity of (1) $\lim_{n\to\infty}\nu_n^{1/n}=\tau(S)$.

The simple case of the circular disk is first studied and complete answers are obtained. Extensions are then made, whenever possible, to more general sets. As an example of the kind of result to be found here, we quote from Theorem 9: Let $A_n(z)=z^n+\cdots$ minimize N over $A_n(\gamma_1, \cdots, \gamma_k)$ with (2) $k=k(n)=O(1)$ and (3) $|\gamma_j|\leqq\binom{n}{j}\alpha_j(n)$, $1\leqq j\leqq k$. Then (1) is valid, provided (4) $\limsup\{\alpha_j(n)\}^{1/n}\leqq 1$, $1\leqq j\leqq\max_{n\geqq 2}k(n)$. Conversely, if (2) holds, then (4) is a necessary condition for (1) provided S is a compact set of regular complement K. In part II, the validity of (5) $\lim_{n\to\infty}|A_n(z, N)|^{1/n}=|\zeta(z)|$ is studied, together with questions relating to the distribution of the zeros of the extremal polynomials. *P. Davis.*

Makar, Ragy H.; and Mikhail, M. N. On induced basic sets of polynomials. Proc. Math. Phys. Soc. Egypt **5** (1956), no. 4, 1–6 (1957). (Arabic summary)

The induced set of a basic set of polynomials was introduced and discussed by Makar [same Proc. **4**, no. 1 (1949), 61–69; no. 2 (1951), 59–64; **5**, no. 1 (1953), 41–46; MR **12**, 689; **13**, 647; **16**, 244]. With the usual notation, let $D(n)=O(n)$ and let the given set be effective in an annulus $0<a\leq R\leq b<\infty$. The authors conjecture that the induced set is then effective in and only in $|z|\leq 1$. They prove this when $\pi_{nl}p_{lj}=0$, except for a finite number of values of j "which are sharply defined in terms of n", but they do not say precisely what the last phrase means.

R. P. Boas, Jr. (Evanston, Ill.).

Mikhail, M. N. The square root set of a simple basic set of polynomials. Duke Math. J. **25** (1957), 177–180.

The author shows that the square-root set of a simple basic set of polynomials (i.e., the set whose product with itself is the original set) is effective on the same circles as the original set, or has the same order as the original set (whichever is appropriate). This considerably extends the results of Nassif and Makar [Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 120–129; MR **16**, 694].

R. P. Boas, Jr. (Evanston, Ill.).

Meschkowski, Herbert. Interpolation durch Funktionen eines Orthonormalsystems. Arch. Math. **8** (1957), 175–179.

Using the scalar product $\int_R fg\,ds=(f,g)$ where the integration runs over the boundary R (composed of finitely many continuous curves) of a schlicht domain $\mathfrak{B}$ of the complex plane, the author considers the (incomplete) Hilbert space $\mathfrak{R}$ of functions analytic in $\mathfrak{B}+R$ and satisfying $(f,f)<\infty$. If $\{\varphi_\nu(z)\}$ is any fundamental orthonormal set in $\mathfrak{R}$, then $K(z,u)=\sum_1^\infty \varphi_\nu(z)\overline{\varphi_\nu(u)}$ converges uniformly in compact subsets of $\mathfrak{B}$. The author constructs an orthonormal set recursively, given a convergent sequence of points $\{u_\nu\}$ which, together with its limit, lies in $\mathfrak{B}$, by requiring that $H_\nu(z)$ be in the linear span of $K(z,u_1),\cdots,K(z,u_\nu)$, and satisfy $H(u_1)=\cdots=H(u_{\nu-1})=0$, $H(u_\nu)=1$, and normalizing the outcome. He then interprets the Fourier expansion $F(z)=\sum_1^\infty b_\nu H_\nu(z)$ as an interpolatory expansion, associated with interpolation at the sequence $\{u_\nu\}$, and observes that $H_\nu(z)$ is orthogonal to all f in $\mathfrak{R}$ which vanish at $u_1,\cdots,u_{\nu-1}$. He notes (but does not solve) the question of determining sufficient conditions on $\{b_\nu\}$ that the interpolation $f(u_\nu)=b_\nu$, $\nu=1,2,3,\cdots$, be solvable in $\mathfrak{R}$ (or even its closure, $H^2(\mathfrak{B})$). The results are all contained in the more complete studies by Walsh and Davis [J. Analyse Math. **2** (1952), 1–28; Trans. Amer. Math. Soc. **76** (1954), 190–206, esp. p. 200; MR **16**, 580; **15**, 803] with which the author is apparently unfamiliar. *A. B. Novikoff* (Berkeley, Calif.).

See also: Sequences, Series, Summability: Gaier. **Difference Equations, Functional Equations:** Boas. **Quantum Mechanics:** Hall and Wightman.

Geometric Analysis

See: Differential Geometry: McConnell. **Elasticity, Plasticity:** Doyle.

Harmonic Functions, Convex Functions

Schiffer, Menahem. Sur la polarisation et la masse virtuelle. C. R. Acad. Sci. Paris **244** (1957), 3118–3121.

The inequalities derived in an earlier paper [same C. R. **244** (1957), 2680–2683; MR **19**, 406] by the same author are used here to prove two inequalities suggested by Pólya [Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 218–221; MR **9**, 111] and the author and Szegö [Trans. Amer. Math. Soc. **67** (1949), 130–205; MR **11**, 515]. One can represent the polarization P in the form $P=4\pi E-V$, where V is the volume enclosed by the boundary S of the problem, and the virtual mass W by $W=4\pi D-V$. For these quantities, the inequalities $EG\geq V$ and $D(4\pi V-G)\geq V$ hold. Here G is the quadratic form belonging to the tensor g_{ik}

$$g_{ik}=\int_S\int_S \frac{\partial x_i}{\partial n_x}\frac{\partial \xi_k}{\partial n_\zeta}\frac{1}{r(z,\zeta)}\,d\sigma_\zeta d\sigma_z,\quad z=(x_i),\ \zeta=(\xi_i),$$

where x_i and ξ_i are rectangular coordinates. From these inequalities, it follows that, for the average polarization p_m and the average virtual mass w_m, the previously conjectured inequalities $p_m\geq 2V$ and $w_m\geq\frac{1}{2}V$ hold. Equality is obtained for the sphere. *U. Hochstrasser.*

Schiffer, Menahem. Problèmes aux limites et fonctions propres de l'équation intégrale de Poincaré et Fredholm. C. R. Acad. Sci. Paris **245** (1957), 18–21.

The theory developed in two previous notes [see the review above and reference therein] is applied to the case of Fredholm's integral equation. Using the smallest positive non-trivial eigenvalue Λ and the largest negative eigenvalue $-\lambda$ of the corresponding eigenvalue problem, one can derive a double inequality for the polarization P and the virtual mass W in any direction:

$$2\pi(1-1/\Lambda)\leq(4\pi V-G)/P\leq 2\pi(1+1/\lambda);$$
$$2\pi(1-1/\lambda)\leq G/W\leq 2\pi(1+1/\Lambda);$$

where V, the volume enclosed by the boundary, and G depend only on the geometry of the problem. From these inequalities one deduces that $\Lambda\leq 3$ for any closed surface. Therefore, if the solution of the integral equation is represented as a Neumann series, this series does not converge more rapidly than a geometric series having the factor $\frac{1}{3}$. λ converges to 1 if the closed surface flattens into a plate. *U. Hochstrasser* (Lawrence, Kans.).

Huber, Alfred. Correction to: "The reflection principle for polyharmonic functions". Pacific J. Math. **7** (1957), 1731.

Revision of the proof in same J. **5** (1955), 433–439 [MR **17**, 28].

Boggio, Tommaso. Sopra un teorema di Almansi relativo al problema biarmonico. Boll. Un. Mat. Ital. (3) **12** (1957), 369–376.

L'A., cui si devono, come é ben noto, fin dal 1901 alcuni dei primi classici risultati sull'esistenza e l'unicità della soluzione del problema al contorno interessante l'equilibrio delle piastre elastiche piane

$$(1)\qquad \Delta\Delta u=0 \text{ in } \Omega;\ u=f \text{ su } \Gamma;\ \frac{\partial u}{\partial n}=g \text{ su } \Gamma$$

(Δ laplaciano, Γ frontiera della regione piana Ω, n normale a Γ), riprende ora il problema (1) e ridimostra, con semplici strumenti di teoria delle funzioni di variabile complessa, un teorema di E. Almansi [Atti Accad. Lincei.

Rend. Cl. Sci. Fis. Mat. Nat. (5) **9** (1900), 1° semestre, 298–304] relativo all'esistenza e al calcolo effettivo di una soluzione regolare (necessariamente unica) di (1), nell'ipotesi che la regione Ω si possa rappresentare conformemente su un cerchio mediante polinomi e che i dati f e g siano sufficientemente regolari. *E. Magenes* (Genova).

Carleson, Lennart. On the connection between Hausdorff measures and capacity. Ark. Mat. **3** (1958), 403–406.

As is known, if the Hausdorff measure of a closed pointset E is >0 for some increasing function $h(r)$, $h(0)=0$, such that $\int_0^\delta (h(t)/t)dt$ converges, then the logarithmic capacity of E is >0 (Frostman; he has proved a more general theorem). The author answers in the negative the converse question: given E of positive capacity, does there exist a measure function $h(r)$ with the above properties such that the corresponding Hausdorff measure is positive? The proof is very concise. *B. Germansky.*

Massera, J. L.; and Schäffer, J. J. Correction to the article "On the level curves of a convex surface". Bol. Fac. Ingen. Agrimens. Montevideo **6** = Fac. Ingen. Agrimens. Montevideo. Publ. Inst. Mat. Estadist. **3** (1957), 65–67. (Spanish. English summary)

In the paper mentioned in the title [Bol. Fac. Ingen. Montevideo **4** (1953), 665–668; MR **15**, 737], the authors had stated a theorem giving necessary and sufficient conditions for the existence of a change of the independent variable t transforming a family of non-decreasing functions $f(t)$ simultaneously into convex functions. In the present note, it is announced that the proof was not correct and that an assumption, viz., the existence of left (or right) limits of the derivatives of the functions $f(t)$, has to be added. Correct versions of the theorem and of the part of the proof in question are given.

W. Fenchel (Copenhagen).

See also: Functions of Real Variables: Dinghas. **Probability:** Hunt.

Special Functions

Chihara, T. S. On co-recursive orthogonal polynomials. Proc. Amer. Math. Soc. **8** (1957), 899–905.

On considère la suite de polynomes $\{P_n(x)\}$ définie par:

$$P_n(x)=(x+b_n)P_{n-1}(x)-\lambda_n P_{n-2}(x)$$
$$(b_n \text{ réel}; \lambda_n>0; n=2, 3, \cdots; P_0=1, P_1=x+b_1)$$

et la suite $P_n^*(x, c)$ définie par cette même formule de récurrence, avec $P_0^*=1$, $P_1^*=P_1-c$, où c est une constante réelle; on a: $P_n^*(x, c)=P_n-cQ_{n-1}$; la suite $\{Q_n(x)\}$ est définie par la formule analogue à la précédente, obtenue en remplaçant b_n et λ_n par b_{n+1} et λ_{n+1}, et en posant: $Q_{-1}=0$, $Q_0=1$. Etude des positions relatives des zéros $x_{n,j}$, $x_{n,j}^*$, $y_{n,j}$ de P_n, P_n^*, Q_{n-1} respectivement ($n\geqq 2$); pour $c>0$, on a:

$$x_{n,j-1}<x_{n,j-1}^*<y_{n,j-1}<x_{n,j}<x_{n,j}^* \quad (j=2, \cdots, n);$$

pour $c<0$, les rôles de $x_{n,j}$ et $x_{n,j}^*$ sont renversés.

Soient $\psi(x)$, $\psi^*(x)$, $\psi_1(x)$ des solutions des problèmes des moments associés à ces trois suites de polynomes; si, pour ψ, le problème des moments correspondant est déterminé, il l'est aussi pour ψ^* et ψ_1, et deux de ces trois fonctions, croissantes au sens large, ne peuvent avoir de pointe de discontinuité commun. Etude de quelques exemples. *J. Favard* (Paris).

Cook, Richard K. Some properties of Struve functions. J. Washington Acad. Sci. **47** (1957), 365–368.

This paper concerns the Struve function $\mathbf{H}_\alpha(z)$, which is closely related to the Bessel function $J_\alpha(z)$. An analogue of the Neumann expansion for $z^{-\alpha}J_\alpha(z)$ is obtained for $z^{-\alpha}\mathbf{H}_\alpha(z)$, valid for arbitrary complex z, provided $\mathrm{Re}\,\alpha>-\frac{1}{2}$. An addition formula is derived for $\mathbf{H}_0(\omega)$, where $\omega^2=z^2+Z^2-2zZ\cos\phi$, which however is much more involved than Graf's formula for $J_0(\omega)$. *C. A. Swanson.*

Ragab, F. M. Integrals involving products of E-functions, Bessel-functions and generalized hypergeometric functions. Boll. Un. Mat. Ital. (3) **12** (1957), 535–551.

In this paper the author proves a general expansion of the integral

$$\int_0^\infty \lambda^{k-1}E(p;\alpha_r:q;\rho_s:\lambda)\,E(l;\beta_t:m;\sigma_u:z/\lambda^2)d\lambda$$

in terms of three of MacRobert's E-functions, under the conditions $p\geqq q+1$, $l\geqq m+1$, $\mathrm{Re}(\alpha_r+k)>0$, $\mathrm{Re}(2\beta_t-k)>0$, and $|\mathrm{amp}\, z|<\pi$. He then deduces a large number of special cases involving generalized hypergeometric functions, confluent hypergeometric functions and Bessel functions. Of these, two of the most interesting are

$$\int_0^\infty \lambda^{k-1}E(\alpha_1, \cdots, \alpha_p:k+1, \rho_1, \cdots, \rho_q:\lambda)$$
$$\times E\left\{\begin{matrix} k+\frac{1}{2}, \frac{1}{2}(k+\rho_1), \cdots, \frac{1}{2}(k+\rho_q), \\ \qquad \frac{1}{2}(\rho_1+1+k), \cdots, \frac{1}{2}(\rho_q+1+k):z/\lambda^2 \\ \frac{1}{2}(\alpha_1+k), \cdots, \frac{1}{2}(\alpha_p+k), \\ \qquad \frac{1}{2}(\alpha_1+1+k), \cdots, \frac{1}{2}(\alpha_p+1+k) \end{matrix}\right\} d\lambda$$
$$=2^{\Sigma(\alpha-\rho)+q-p-k+1}\pi^{\frac{1}{2}(q-p+1)}z^{\frac{1}{2}k}J_k[(4^{1+p-q}/z)^{\frac{1}{2}}]K_k[(4^{1+p-q}/z)^{\frac{1}{2}}]$$

if $p\geqq q+1$, $\mathrm{Re}(\alpha_r+k)>0$, $\mathrm{Re}(\rho_s)>0$, $|\mathrm{amp}\, z|<\pi$ and $\mathrm{Re}(k)>-1$, and a similar result involving

$$J_{k-1}[(4^{1+p-q}/z)^{\frac{1}{2}}]K_{k-1}[(4^{1+p-q}/z)^{\frac{1}{2}}].$$

L. J. Slater (Cambridge, England).

See also: General Theory of Numbers: Erdös. **Functions of Complex Variables:** Stallmann. **Approximations, Orthogonal Functions:** Brown. **Numerical Methods:** Forsythe. **Optics, Electromagnetic Theory, Circuits:** Feuerstein. **Quantum Mechanics:** Meligy.

Sequences, Series, Summability

Barna, Béla. Über ein Iterationsverfahren mit zwei Variablen. Acta Univ. Debrecen **3** (1956), no. 2, 13–21 (1957). (Hungarian. German summary)

Der iterative Algorithmus $a_{n+1}=\frac{1}{2}(a_n+(a_nb_n)^{\frac{1}{2}})$, $b_{n+1}=\frac{1}{2}(b_n+(a_nb_n)^{\frac{1}{2}})$ ($n=0, 1, 2, \cdots$; $a_0=a$, $b_0=b$; a, b beliebig complex) wird untersucht. Es wird bewiesen, dass beide Folgen (a_n) und (b_n) demselben Grenzwert $M(a, b)$ zustreben. Ausser in dem trivialen Fall $ab(a-b)=0$, hat $M(a, b)$ unendlich viele Werte, unter denen immer die Zahl 0 vorkommt; für alle anderen Werte gilt $M(a, b)=(b-a)/\mathrm{Log}(b/a)$, wo $\mathrm{Log}(b/a)$ ein passender Wert des unendlich vieldeutigen komplexen Logarithmus ist.

Aus der Zusammenfassung des Autors.

Devinatz, A. Two parameter moment problems. Duke Math. J. **24** (1957), 481–498.

The author considers the problem whether the natural two-parameter analogues of the classical conditions of Stieltjes and Hamburger for the one-parameter moment-

problems are sufficient in the two-parameter case. (The general theorem of Haviland which can be applied to derive solutions of various classical moment-problems [cf. Shohat and Tamarkin, The problem of moments, Amer. Math. Soc. Math. Surveys, v. 1, New York, 1943; MR **5**, 5; **13**, 1138] seems to yield nothing here.) The author gives a solution of this problem under an additional restriction which amounts to demanding uniqueness of the measures giving the representations. His result is as follows: If $\{\mu(m, n)\}(m, n=0, 1, \cdots)$ is a sequence of real numbers such that (a) $\mu(m+p, n+q)\geqq 0$, (b) $a_i\mu(m+p, n+q)\leqq \mu(m+p+2-i, n+q+i-1)\leqq b_i\mu(m+p, n+q)$, $i=1, 2$, and (c) for every fixed m_0, n_0, the Hamburger sequences $\{\mu(m, 2n_0)\}$ and $\{\mu(2m_0, n)+\mu(0, n)\}$ are determined, then there exists a uniquely determined bounded measure $d\alpha(x, y)\geqq 0$ such that $\mu(m, n)=\int_{a_2}^{b_2}\int_{a_1}^{b_1} x^m y^n d\alpha$. The condition (b), or (c), may be removed if $a_k=-\infty$ ($b_k=\infty$), or if $-\infty<a_k<b_k<\infty$. Here, $\mu(m+p, n+q)\geqq 0$ means that $\sum_i\sum_j z_i z_j \mu(m_i+m_j, n_i+n_j)\geqq 0$ for every finite set of real numbers $\{z_i\}$ and points $\{(m_i, n_i)\}$; $\mu\geqq\nu$ means $\mu-\nu\geqq 0$. The proof uses methods, based on the theory of reproducing kernels and operators in Hilbert spaces, already applied by the author in his previous papers on moment-problems. An appendix contains an interpretation and proof of a result of Hamburger in the Hilbert space setting which is used in the paper. *M. Cotlar.*

Jakimovski, A. Some remarks on the moment problem of Hausdorff. J. London Math. Soc. **33** (1958), 1–14.

Let $\{\mu_n\}$ be a sequence of real numbers, $\Delta^1\mu_n=\mu_{n+1}-\mu_n$, $\Delta^k\mu_n=\Delta^1\Delta^{k-1}\mu_n$, $\lambda_{k,m}=(-1)^{k-m}\binom{k}{m}\Delta^{k-m}\mu_m$ $(0\leqq m\leqq k)$ and $\|a_{n,k}\|$ a regular summability matrix. The author shows that $\mu_n=\int_0^1 t^n d\alpha(t)$, with $\alpha(t)$ of bounded variation, if and only if $\sum_{k=0}^{\infty}|a_{n,k}|\sum_{m=0}^{k}|\lambda_{k,m}|<L<\infty$ uniformly for all $n\geqq 0$. This is a weaker form of the well-known conditions of Hausdorff.

Let $\{t_n\}$ and $\{s_n\}$ be sequences connected by the transform $t_n=\sum\lambda_{n,m}s_m$ and $\{t_{n_i}\}$ a fixed subsequence of $\{t_n\}$. As an application of the above result it is shown that $\{t_{n_i}\}$ converges whenever $\{s_n\}$ converges, and to the same limit, if and only if the same is true for the full sequence $\{t_n\}$.

If $P(x)=\sum_0^n a_m x^m$, set $M\{P\}=\sum_0^n a_m\mu_m$, where $\{\mu_m\}$ is a fixed sequence of numbers. If $B_k[f(x)]$ is the kth Bernstein polynomial of f, it is shown that $\mu_n=\lim_{k\to\infty} M\{B_k[x^n]\}$. This generalizes a known result [D. V. Widder, The Laplace transform, Princeton, 1941, p. 102; MR **3**, 232] where it is assumed $\{\mu_n\}$ is a moment sequence.

A sequence $\{\mu_n\}$ may be written as $\mu_n=\int_0^1 t^{n+1}d\alpha(t)$ $(n\geqq p\geqq 0)$, $\alpha(t)$ of bounded variation, if and only if for some infinite non-decreasing sequence of integers $\{q_i\}_0^{\infty}$ with $q_0=p$ and $q_{i+1}-q_i\leqq p+1$, $\sum_{m=q_i}^{\infty}|\lambda_{m,q_i}|<L<\infty$ holds uniformly for all i. This generalizes a result of M. S. Ramanujan [J. Indian Math. Soc. (N.S.) **17** (1953), 47–53; MR **15**, 118]. The author also proves a generalization of this result and gives an application to "quasi-Hausdorff transformations".

Some final results give a connection between "Boole's polynomials" and the Hausdorff moment problem.

A. Devinatz (St. Louis, Mo.).

Siegel, Carl Ludwig. Vereinfachter Beweis eines Satzes von J. Moser. Comm. Pure Appl. Math. **10** (1957), 305–309.

Let an area-preserving hyperbolic mapping M be given by two real power series in x and y (assumed to converge in a neighborhood of the origin), $x_1=\lambda x+ax^2+bxy+cy^2+\cdots$, $y_1=\lambda^{-1}y+Ax^2+Bxy+Cy^2+\cdots$, with $|\lambda|>1$. Then there exists an area-preserving substitution (in the form of two formal power series in ξ and η), $x=\phi(\xi, \eta)$, $y=\psi(\xi, \eta)$, which transforms M into the normal form $\xi_1=U(\xi\eta)\xi$, $\eta_1=U^{-1}(\xi\eta)\eta$, with $U(\xi\eta)$ a power series in $\xi\eta=\omega$, convergent in the neighborhood of $\omega=0$. It was shown by J. Moser [same Comm. **9** (1956), 673–692; MR **19**, 278] that the substitution mentioned above can actually be so chosen that the power series ϕ and ψ are convergent in a neighborhood of the origin. In his proof, Moser reduced this convergence problem to one of a canonical system of differential equations. The present paper gives a direct proof of Moser's theorem by applying the majorant method to ϕ and ψ themselves.

F. Herzog (East Lansing, Mich.).

Gaier, Dieter. Eine Bemerkung zum unstetigen Abel-Verfahren. Arch. Math. **8** (1957), 286–289.

Die Reihe $\sum a_n$ heißt A-summierbar, wenn $f(z)=\sum a_n z^n$ in $|z|<1$ regulär ist und $\lim_{x\to 1-0} f(x)$ existiert. Verlangt man hier nur die Existenz vom $\lim_{n\to\infty} f(1-n^{-1})$, so erhält man das Verfahren A_1. Verf. gibt einen überraschend scharfen Vergleichssatz für diese beiden Verfahren: (a) Für Reihen mit $a_n=O(e^{c\sqrt{n}})$ (wo $c<\sqrt{(2\pi)}$) sind die beiden Verfahren äquivalent; (b) für $c=\sqrt{(2\pi)}$ wird die Aussage (a) falsch. Zum Beweis von (a) betrachtet Verf. die Funktion $f(w/(w+1))$, die unter den genannten Bedingungen für die a_n in einem Winkelraum vom Exponentialtyp ist, so daß die Existenz von $\lim f(1-n^{-1})$ die von $\lim f(x)$ nach sich zieht. Als Beispiel für (b) dient $f(z)=(1-z)^{-1}\sin(\pi z/(1-z))$.

K. Zeller (Tübingen).

Russell, D. C. Note on inclusion theorems for infinite matrices. J. London Math. Soc. **33** (1958), 50–62.

For the main result (Theorem 1), it is required that $A=(a_{n,k})$ shall be normal (a lower semi-matrix with non-zero diagonal elements, i.e., $a_{n,k}=0$ $(k>n)$, $a_{n,n}\neq 0$), that it shall have zero column limits, and that it satisfies a mean value condition, denoted by $M(A)$, of the following type: For each sequence $\{S_k\}$ of a given class, and for all $n\leqq m$, there exists n' with $0\leqq n'\leqq n$ and an absolute constant K such that

$$M(A): \qquad \left|\sum_{k=0}^{n} a_{m,k}s_k\right|\leqq K\left|\sum_{k=0}^{n'} a_{n',k}s_k\right|.$$

For Cesàro means (C, κ), $0<\kappa\leqq 1$, this was first stated without proof by Jacob [Proc. London Math. Soc. (2) **26** (1927), 470–492], and proved independently by Bosanquet [J. London Math. Soc. **16** (1941), 146–148; MR **3**, 144]; a stronger version of the inequality appears in an alternative proof by Bosanquet [Proc. London Math. Soc. (2) **50** (1949), 482–496, Lemma 5; MR **10**, 368]. This method was adapted by Jurkat and Peyerimhoff [Math. Z. **55** (1951), 92–108, Satz 1; MR **13**, 934] to prove the following conditions, dependent only on the matrix elements: It is sufficient, for $M(A)$ to be satisfied for every sequence $\{s_k\}$, that $a_{n,k}\neq 0$, $0\leqq a_{m,k}/a_{n,k}\leqq K$ $(0\leqq k\leqq n\leqq m)$, and $a_{m,k-1}/a_{n,k-1}\geqq a_{m,k}/a_{n,k}$ $(1\leqq k\leqq n\leqq m)$.

The author leads, through a series of lemmas and corollaries, to the main result.

Theorem 1. Let A be normal and satisfy $M(A)$ for every A-limitable sequence $\{s_n\}$, and let $\lim_{n\to\infty} a_{n,k}=0$ for every fixed k. Let C be an arbitrary matrix. The necessary and sufficient conditions that the existence of A-$\lim s_n$

implies that of C-lim s_n are that a matrix $(b_{n,\nu})$ and a sequence $\{\theta_n\}$ exist satisfying $b_{n,\nu}=\sum_{k=\nu}^{\infty} c_{n,k}a_{k,\nu}^{-1}$, $\theta_n=\sum_{k=0}^{\infty} c_{n,k}\sum_{\nu=0}^{k} a_{k,\nu}^{-1}$, and such that $c_{n,k}=\sum_{\nu=k}^{\infty} b_{n,\nu}a_{\nu,k}$, $\sum_{\nu=0}^{\infty}|b_{n,\nu}|<M$ for every n, $\lim_{n\to\infty} b_{n,\nu}=\beta_\nu$ for each fixed ν, and $\lim_{n\to\infty}\theta_n=\theta$. Moreover, if A-lim $s_n=\sigma$, then C-lim $s_n=\theta\sigma+\sum_{\nu=0}^{\infty}\beta_\nu(\sigma_\nu-\sigma)$, where $\sigma_\nu=\sum_{n=0}^{\infty} a_{\nu,n}s_n$. The condition $\lim_{n\to\infty}\theta_n=\theta$ is to be omitted if $\sigma=0$.

Theorem 1A. Let A be a multiplicative matrix with non-zero multiplier, normal, and satisfy $M(A)$ for every A-limitable sequence $\{s_n\}$; let C be an arbitrary matrix. Then the existence of A-lim s_n implies that of C-lim s_n if, and only if, there is a K-matrix B such that $BA=C$. Moreover, if A-lim $s_n=\sigma$ and B has characteristic numbers β, β_ν, then C-lim $s_n=\beta\sigma+\sum_{\nu=0}^{\infty}\beta_\nu(\sigma_\nu-\sigma)$.

Theorem 1B. Let A be a normal T-matrix satisfying $M(A)$ for every A-limitable sequence $\{s_n\}$, and let B be an arbitrary T-matrix. Then $(BA)\geq A$. ($C\geq A$ means that A-lim $s_n=\sigma$ implies C-lim $s_n=\sigma$.)

Theorem 2. Let A, C be row-finite matrices, and let A possess a row-finite left-hand reciprocal ${}^{-1}A$. Then the necessary and sufficient condition that C-lim s_n exists whenever A-lim s_n exists is that $C\cdot{}^{-1}A$ should be a K-matrix. (If the limits are to be the same, we see that the necessary and sufficient condition that $C\geq A$ is that $C\cdot{}^{-1}A$ should be a T-matrix; this is then similar to a result of the reviewer and A. M. Barnett [J. London Math Soc. 23 (1948), 211–221, Th. III; MR 10, 447], whose theorem would then contain the sufficiency part of the author's Theorem 2.)

Finally, application is made of the above to Nörlund summability. *R. G. Cooke* (London).

Keogh, F. R.; and Petersen, G. M. A universal Tauberian theorem. J. London Math. Soc. 33 (1958), 121–123.

Some years ago, the reviewer showed that a sequence $\{s_n\}$ is convergent if there is a regular matrix summability transform T which sums all its subsequences. [Bull. Amer. Math. Soc. 49 (1943), 898–899; Proc. Amer. Math. Soc. 7 (1956), 1074–1075; MR 5, 117; 18, 478]. Using a slight modification of the original proof, the authors of this paper show that it is sufficient for T to sum a set of subsequences of $\{s_n\}$ which is of second category in the natural mapping onto [0, 1]. Accordingly, they also observe that if $\{s_n\}$ is any divergent sequence which is bounded and summable $(C, 1)$, then the subsequences which are also summable $(C, 1)$ correspond to a set S on [0, 1] which is of first category, but, by another result of the reviewer and H. Pollard [Bull. Amer. Math. Soc. 49 (1943), 924–931; MR 5, 117] is of measure 1. *R. C. Buck.*

Keogh, F. R.; and Petersen, G. M. A generalized Tauberian theorem. Canad. J. Math. 10 (1958), 111–114.

For notation and references see the preceding review. In the present note, the authors show that if enough (=second category) subsequences of $\{s_n\}$ have T transforms with kernel of diameter less than d, then the same must hold for $\{s_n\}$ itself. [See also Agnew and Hill, Duke Math. J. 11 (1944), 573–574; MR 6, 46.]

R. C. Buck (Madison, Wis.).

See also: Functions of Complex Variables: Kaplan. **Special Functions:** Chihara. **Approximations, Orthogonal Functions:** Gregorini. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Krabbe; Day.

Approximations, Orthogonal Functions

Brown, J. L., Jr. A simplified derivation of the Fourier coefficients for Chebyshev patterns. Inst. Elec. Engrs. Monograph no. 265R (1957), 2 pp.

The author considers the problem of determining the coefficients b_m in the expansion $T_n(ax+b)=\sum_{m=0}^{n} b_m T_m(x)$ where a, b are constants, $n>0$, and $T_m(x)=\cos(m \arccos x)$, $m\geq 0$, is a Chebyshev polynomial of degree m. Salzer [Proc. Inst. Elec. Engrs. C. 103 (1956), 286–288; MR 19, 136] developed a formula for the b_m by means of several series expansions. The present paper derives the same formula using a method involving the orthogonality properties of the Chebyshev polynomials.

D. M. Young, Jr. (Los Angeles, Calif.).

Gregorini, Giovanna. Un'osservazione sui coefficienti di Legendre-Stieltjes di una funzione non decrescente. Boll. Un. Mat. Ital. (3) 12 (1957), 655–657.

Ghizzetti [Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 753–758; MR 19, 545] gave determinantal conditions for a sequence of numbers to be Legendre-Stieltjes coefficients of a nondecreasing function. In one of Ghizzetti's cases, the nondecreasing function is a step-function with a finite number of jumps, four arrangements of the jumps being possible. The author gives criteria for distinguishing among these possibilities.

R. P. Boas, Jr. (Evanston, Ill.).

Butzer, Paul L. Sur la meilleure approximation d'une fonction définie par sa transformée de Laplace. C. R. Acad. Sci. Paris 246 (1958), 687–689.

The Laplace integral of a function in $L^p[0, \infty)$, $1\leq p<\infty$, is inverted by means of the Fourier-Laguerre series, and the degree of approximation of certain partial sums are obtained. The same question is also treated via Hermite series. *P. Civin* (Eugene, Oreg.).

Davis, Harry F. A note on asymptotic series. Canad. J. Math. 9 (1957), 90–95.

Let C denote the collection of all asymptotic sums of formal power series without negative exponents. The notion of a continuous linear functional on C is introduced and it is shown that such functionals have a simple form. The notion of an asymptotically continuous linear functional on a subspace of C is introduced. A characterization is given for all such functionals on the subspace consisting of all finite linear combinations of x^m ($m=0, 1, 2, \cdots$) and e^{-tz} ($t\geq 0$). Other results are given including a proof that the two notions referred to above are distinct, at least over proper subspaces. *T. E. Hull.*

See also: Functions of Complex Variables: Fekete and Walsh; Makar and Mikhail; Mikhail; Meschkowski.

Trigonometric Series and Integrals

Efimov, A. V. Approximation of certain classes of continuous functions by Fourier sums and Fejer means. Dokl. Akad. Nauk SSSR (N.S.) 114 (1957), 930–933. (Russian)

The author considers classes $W^r{}_\beta H_k{}^\alpha$ ($r\geq 0$, $\alpha\geq 0$, $k=0, 1, 2, \cdots$) of functions which have the Fourier series of the form

$$\sum_{m=1}^{\infty} \pi^{-1}m^{-r}\int_{-\pi}^{\pi}\varphi(x+t)\cos(mt+\tfrac{1}{2}\beta\pi)dt,$$

where the function $\varphi(x)$ belongs to the class H_k^α, i.e., the differences $\Delta_\delta^k\varphi(x)=\sum_{i=0}^{k}(-1)^{k-i}\ \varphi(x+\frac{1}{2}\delta(k-2i))$ satisfy the inequality $|\Delta_\delta^k\varphi(x)|\leqq\delta^\alpha$, $-\pi\leqq x\leqq\pi$. $f\in\mathfrak{M}$ means that the conjugate function $\bar{f}$ belongs to the class $\mathfrak{M}$. Asymptotic expressions for differences $f(x)-U_nf(x)$ for functions f of a certain class $\mathfrak{M}$ and for the degree of the uniform approximation $\mathscr{E}_{U_n}(\mathfrak{M})=\sup_{f\in\mathfrak{M}}\|f-U_nf\|$ are sought. In the case in which the operator U_nf is the partial sum S_n of the Fourier series of f or the Fejèr sum σ_n, this problem was studied among others by Nikolskiĭ [Trudy Mat. Inst. Steklov. **15** (1945); MR **7**, 435], Zamansky [Ann. Sci. Ecole Norm. Sup. (3) **66** (1949), 19–93; MR **11**, 27], and Stečkin [Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 643–648; MR **18**, 303]. The author announces several further results in this direction, for example $\mathscr{E}_{\sigma_n}(\bar{H}_2^1)=\log n/[2n\log(2^1+1)]+O(n^{-1})$, and $\mathscr{E}_{S_n}(W_\beta^rH_k^\alpha)=C_k(\alpha)\pi^{-1}\log n/n^{r+\alpha}+O(\log\log n/n^{r+\alpha})$, where $C_k(\alpha)=\sup_{f\in H_k^\alpha}|\pi^{-1}\int_{-\pi}^{+\pi}f(x)\cos x\,dx|$.

G. G. Lorentz (Detroit, Mich.).

Keogh, F. R. A theorem on summability of Fourier-Stieltjes series. J. London Math. Soc. **32** (1957), 507–510.

The author essentially gives a sufficient condition for the summabitily of a Fourier-Stieltjes series by a "kernel" which includes the Abel and Cesàro kernels.

K. Chandrasekharan (Bombay).

Erdös, P.; and Taylor, S. J. On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences. Proc. London Math. Soc. (3) **7** (1957), 598–615.

This paper consists of three parts.

I. If $0\leqq\mu_k\leqq2\pi(k=1, 2, \cdots)$, and (n_k) is an increasing sequence of integers such that $1<t_k=n_{k+1}/n_k\leqq K<\infty$, then $\sin(n_kx-\mu_k)\to0$ as $k\to\infty$ for at most countable x. If $1<\rho\leqq t_k\leqq K<\infty$, then $\sin(n_kx-y)\to0$ as $k\to\infty$ for at most enumerable pairs (x, y) $(0\leqq x\leqq2\pi,\ 0\leqq y\leqq2\pi)$; and hence there is an at most enumerable linear set Q such that $\sum\sin(n_kx-y)$ does not converge for any x unless $y\in Q$. In the case $t_k\to\infty$, the set of x such that $\sum\sin(n_kx-\mu_k)$ converges has dimension 1.

II. If t_k is an integer for sufficiently large k and $t_k\to\infty$, then the set of x such that $\sum\sin n_kx$ converges absolutely has the power of the continuum. The condition that t_k is an integer is necessary. Further if $\sum t_k^{-1}$ converges, then for any (μ_k) $\sum\sin(n_kx-\mu_k)$ converges absolutely for x in a set of power of the continuum. Suppose that $\lambda k^\rho\leqq t_k\mu k^\rho$ for fixed λ, μ and ρ. If $0<\rho\leqq1$, then the set of x for which $\sum\sin(n_kx-\mu_k)$ converges absolutely has zero dimension; if $\rho>1$, it has $(1-1/\rho)$ dimension.

III. It is well known that, if (n_k) is an increasing sequence of integers, then the set E of x such that the sequence (n_kx) (mod 1) is not equidistributed in (0, 1) has zero Lebesgue measure, and that if n_k is a polynomial in k with integral coefficients, then E is enumerable. If (n_k) is an increasing sequence of integers such that $n_{k+1}-n_k<$constant, then E is not necessarily enumerable, but has dimension zero. If $n_k<Ck^\rho$, C and ρ being constants, then E has dimension $\leqq1-1/\rho$ and if $t_k\geqq\rho>1$, then E has dimension 1. For the proof of some of these theorems, Eggleston's theorem is used [same Proc. (2) **54** (1952), 42–93; MR **14**, 23]. *S. Izumi.*

Kahane, Jean-Pierre. Généralisation d'un théorème de S. Bernstein. Bull. Soc. Math. France **85** (1957), 221–229.

Let E be a closed subset belonging to T, the real numbers modulo 2π, and let E_h be the union of all segments of length $2h$ whose centers belong to E. Let $\mu=\mu(E)$ be the lower bound of those ν for which $|E_h|=O(h^{1-\nu})$ as $h\to0+$. In the present paper it is shown that if $f(x)$ is defined on E and satisfies there a Lipschitz condition of order $\alpha>\frac{1}{2}\mu$, then it is possible to extend $f(x)$ from E to T in such a way that the extended function will have an absolutely convergent Fourier series. (The theorem of Bernstein which asserts that if $f(x)$ defined on T satisfies a Lipschitz condition of order $\alpha>\frac{1}{2}$, then $f(x)$ has an absolutely convergent Fourier series, appears as the special case $E=T$). In point of fact, a suitable extension is obtained by linear interpolation on the open intervals comprising $T-E$; the demonstration is an ingenious modification of Bernstein's original proof. Much more difficult is the construction of several examples which show that if $\alpha<\frac{1}{2}\mu$, then the conclusion may fail. This construction is based on the following interesting principle: if $f(x)=\sum_{-\infty}^{\infty}a_ne^{inx}$, where $\sum_{-\infty}^{\infty}|a_n|=A$ and if $g(x)=\sum_{k=1}^{K}b(k)e^{it_kx}$ (the t_k are not assumed to be integers) satisfies $|g(x)|\leqq M$, $-\infty<x<\infty$, then $|\sum_{k=1}^{K}b(k)f(t_k)|\leqq AM$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

See also: Topological Vector Spaces: Lighthill. **Astronomy:** Sen.

Integral Transforms

Boxer, R.; and Thaler, S. A simplified method of solving linear and nonlinear systems. Proc. I.R.E. **44** (1956), 89–101.

In the mathematical theory of electrical engineering the function $F^*(z)=\sum_{n=0}^{\infty}f(nT)z^{-n}$ is frequently called the "z-transform" of $f(t)$. If $f(t)$ is unknown but its Laplace transform $F(S)$ is a given rational function the authors describe an algebraic algorithm that yields an approximation to $F^*(z)$, and hence to $f(nT)$, for small T. Since the Laplace transform of the solution of initial value problems for linear differential equations with constant coefficients is frequently an immediately available rational function, this method can be employed for the approximate solution of such problems. For linear initial value problems with variable coefficients, and for nonlinear problems the authors propose a modification of their algorithm which amounts essentially to a replacement of the given problem by one that is stepwise linear with constant coefficients. The arguments are partly heuristic. *W. Wasow* (Madison, Wisc.).

Wasow, Wolfgang. Discrete approximations to the Laplace transformation. Z. Angew. Math. Phys. **8** (1957), 401–417.

The author treats the transformation

$$(*)\qquad h\mathfrak{L}_hf=h\sum_{n=0}^{\infty}f(nh)e^{-snh}$$

as an approximation to the Laplace transformation. He calls (*) the z-transformation.

Transformation (*) was treated by T. Fort [Amer. Math. Monthly **62** (1955), 641–645; MR **17**, 860] and J. R. Hanna [ibid. **64** (1957), 576–581; MR **19**, 543] under the name "Dirichlet Series Transform". These papers seem to have been overlooked by the author of the paper under review. At the same time it is to be noted that the two authors mentioned, in addition to a study of the transformation itself, had in mind the solution of the difference equation with discrete independent variable while the

present author is interested in the problem of approximating the solution of a differential equation.

In the paper under review, the author shows that if $f(t)$ is entire and of exponential type, and if it is expanded by Maclaurin's series the z-transformation can be applied termwise. With the use of this result he arrives at Rule 1 of procedure, which, however, is too long to quote in this review. The error in substituting the z-transformation is studied by means of the Euler-Maclaurin sum-formula. Application to ordinary linear differential equations with constant coefficients leads to a second Rule of procedure again too long and detailed to be quoted here. The author throughout makes frequent reference to a paper by R. Boxer and S. Thaler [reviewed above] .It seems to have been this paper which largely inspired the one under review. *T. Fort* (Columbia, S.C.).

Buschman, R. G. A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel. Pacific J. Math. **7** (1957), 1529–1533.

For integral transforms with symmetric kernel, $F(t)$ mapped into $f(s)=\int_a^b K(s,t)F(t)dt$, the author proves a substitution theorem for the transform of $k(t)=F[g(t)]$. He illustrates the result for certain special substitutions in Laplace, Stieltjes and Hankel transforms.

S. Katz.

Fox, Charles. A generalization of the Cauchy principal value. Canad. J. Math. **9** (1957), 110–117.

Es sei $a<u<b$ und n eine ganze Zahl >0. Der Verf. definiert den Hauptwert P des folgenden uneigentlichen Integrales in Verallgemeinerung einer Definition von Cauchy/Hadamard:

$$P\int_a^b \frac{f(x)dx}{(x-u)^{n+1}} = \lim_{\varepsilon\to+0}\left\{\left(\int_a^{u-\varepsilon}+\int_{u+\varepsilon}^b\right)\frac{f(x)dx}{(x-u)^{n+1}}-H_n(u,\varepsilon)\right\}$$

wobei

$$H_0(u,\varepsilon)=0,$$

$$H_n(u,\varepsilon)=\sum_{i=0}^{n-1}\frac{f^{(i)}(u)}{i!}\left\{\frac{1-(-1)^{n-i}}{(n-i)\varepsilon^{n-i}}\right\},\quad n>0.$$

Der Verf. gibt einfache Bedingungen für $f(x)$, damit die rechte Seite des obigen Gl. konvergiert, und zeigt Anwendungen dieser Sätze auf die Hilbert'sche Integral-Transformation und die Formel von Plemelj. *W. Saxer.*

See also: Functions of Complex Variables: Kaplan. **Sequences, Series, Summability:** Devinatz; Jakimovski. **Approximations, Orthogonal Functions:** Butzer. **Topological Vector Spaces:** Lighthill.

Ordinary Differential Equations

Bellert, S. On foundations of operational calculus. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 855–858, LXXIII–LXXIV. (Russian summary)

The author gives a formal description of the operational calculus for various operators (differential, difference, etc.). No discussion of justifications for the procedure is given. *J. L. B. Cooper* (Cardiff).

Pachale, Helmut. Ein Satz über die stetige Differenzierbarkeit der Lösungen von $F(x,y,y')=0$. Elem. Math. **13** (1958), 38–39.

Merli, Luigi. Sul teorema di unicità per il problema di Cauchy relativo alla equazione differenziale

$$x^{(n)}=f(t,x,x^{(1)},\ldots x^{(n-1)}).$$

Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 580–584.

Wintner hat unlängst [Boll. Un. Mat. Ital. (3) **11** (1956), 496–498; MR **19**, 32] den Eindeutigkeitssatz von Nagumo verallgemeinert, indem er ihn für die Gleichung $x^{(n)}=f(t,x)$ abgeleitet hat. Der Verfasser beweist den Satz für die Gleichung $x^{(n)}=f(t,x^{(1)},\cdots,x^{(n-1)})$.

M. Zlámal (Brno).

Opial, Z. Sur l'allure asymptotique des intégrales de l'équation différentielle $u''+a(t)u'+b(t)u=0$. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 847–853, LXXIII (Russian summary)

Der Verfasser beschäftigt sich mit den asymptotischen Eigenschaften der Lösungen von (1) $u''+a(t)u'+b(t)u=0$. Es handelt sich um zwei Fälle. Im ersten Fall strebt jede Lösung von (1) gegen Null bei $t\to\infty$. Im zweiten existieren zwei linear unabhängige Lösungen, von denen eine gegen Null und die andere gegen einen von Null verschiedenen Grenzwert strebt. Der Verfasser gibt Bedingungen an, welche hinreichend und auch notwendig sind, dass der erste oder zweite Fall eintrete. Ein Teil seiner Resultate stellt eine Verallgemeinerung der Resultate des Rezensenten [Czechoslovak Math. J. **6(81)** (1956), 75–93; MR **18**, 210] vor. *M. Zlámal* (Brno).

Wintner, Aurel. On linear perturbations. Quart. Appl. Math. **15** (1958), 428–430.

Es seien $u(t)$, $v(t)$ linear unabhängige Lösungen von (1) $\ddot{x}+f(t)x=0$. Unter der Voraussetzung $\int^\infty |f-g|(|u|^2+|v|^2)dt\to\infty$ wird für jede Lösung $y(t)$ von $\ddot{y}+g(t)y=0$ die asymptotische Formel $y(t)=c_1u(t)+c_2v(t)+o(|u(t)|+|v(t)|)$ bewiesen, wobei c_1 und c_2 Integrationskonstanten sind. Die Formel kan man differenzieren, d.h. es gilt $y'(t)=c_1u'(t)+c_2v'(t)+o(|u'(t)|+|v'(t)|)$. Solche Formel war bisher bekannt nur unter Voraussetzungen, welche (1) durch gewisse Stabilitätsbedingungen beschränken.

M. Zlámal (Brno).

Makai, E. A class of systems of differential equations and its treatment with matrix methods. I. Publ. Math. Debrecen **5** (1957), 5–37.

Let A be a constant square matrix of order n with complex elements, and X a diagonal matrix with diagonal elements $x-a_i$, $i=1, 2, \cdots, n$, and $a_i\neq0$. Consider the differential system (1) $Xy'=Ay$. Among other properties of the system (1), the author points out the following. (i) Let $P_n(x)$ be a polynomial of degree n having simple zeros $a_i\neq0$, and P_{n-1} and P_{n-2} polynomials of degree not exceeding $n-1$ and $n-2$ respectively. Every solution of the second-order differential equation (2) $P_ny''+P_{n-1}y'+P_{n-2}y=0$ is the first component of some solution of an appropriate system (1). It follows that solutions of $y''+f_1y'+f_2y=0$ can be uniformly approximated with arbitrary accuracy by the first components of solutions of appropriate differential systems of type (1) near the origin where $f_1(x)$ and $f_2(x)$ are regular. (ii) If the non-negative integer r is a characteristic root of A, and if no smaller non-negative integers are characteristic roots, then the system (1) has a polynomial solution of degree r. Methods of constructing polynomial solutions are given.

Choy-tak Taam (Washington, D.C.).

Atkinson, F. V. Estimation of an eigen-value occurring in a stability problem. Math. Z. **68** (1957), 82–99.

Let $p(x)$ be real-valued and continuous, $p(x)\not\equiv 0$. Consider the eigen-value problem of the differential equation $d^{2k}y/dx^{2k}=(-1)^k\lambda p(x)y$ under the condition $y(x+2\pi)=\exp(2\pi i\mu)y(x)$, k and μ being fixed, $k=1, 2, \cdots$, $0<\mu\leq\frac{1}{2}$. The author establishes lower bound and upper bound for the least positive eigen-value for each of the following cases: (i) non-negative $p(x)$; (ii) periodic $p(x)$ with period 2π and non-negative integral from 0 to 2π; (iii) odd periodic $p(x)$ with period 2π. These bounds are expressed in terms of the complex Fourier coefficients of $p(x)$; they also fix the order of magnitude of the least positive eigen-value and its asymptotic behaviour for small μ. For $k=1$, these results yield stability criteria, including those of Lyapunov and others. The method of proof is to reduce the eigen-value problem to an infinite system of linear equations which in turn define an eigen-value problem of a completely continuous Hermitean operator in a separable Hilbert space. *Choy-tak Taam.*

Reid, William T. A comparison theorem for self-adjoint differential equations of second order. Ann. of Math. (2) **65** (1957), 197–202.

In Ann. of Math. (2) **64** (1956), 505–513 [MR **19**, 1058], the reviewer gave as an application of his result on partial differential equations a comparison theorem for ordinary differential equations. Namely, if M denotes the maximum distance between consecutive zeros in an interval I_x of a solution of $(r(x)u(x)')'+ru=0$, m the minimum distance between zeros in an interval I_y of a solution of $(s(y)v(y)')'+sv=0$, and if $s'(y)/s(y)\geq|r'(x)/r(x)|$ for $x\in I_x$, $y\in I_y$, then $m/M\geq\frac{1}{2}$. The author shows that this theorem can be proved in a somewhat more general form by a method involving only the consideration of ordinary differential equations. It is shown to be a consequence of a new comparison theorem on focal points derived in the present paper.

By considering the special case where either $r(x)$ or $s(y)$ is an exponential, so that either m or M is known, the author finds oscillation and non-oscillation criteria which generalize results of R. L. Potter [Pacific J. Math. **3** (1953), 467–491; MR **15**, 32]. *H. F. Weinberger.*

Feller, William. Generalized second order differential operators and their lateral conditions. Illinois J. Math. **1** (1957), 459–504.

Let m be a (finite or infinite) regular positive measure on a real interval I, vanishing on no subinterval, and let φ be an everywhere positive continuous function. The author studies the operator $A_\varphi f=\varphi^{-1}D_m\varphi^2D_x\varphi^{-1}f$, continuing the work on second-order operators that he originally undertook in connection with diffusion processes [cf. Ann. of Math. (2) **61** (1955), 90–105; MR **16**, 824]. The differentiation D_m is in the sense of Radon-Nikodym. The operator A_φ is considered to be densely defined either on the space $L^2(m, I)$ of m-square-integrable functions, or else on the space $C(I)$ of continuous functions. In L^2, when m is finite, Feller allows the endpoints of I to carry positive weights, and in so doing defines conditions on A_φ from which the classical boundary conditions are obtained as a degenerate limiting case. For particular absolutely continuous m and suitable function-multipliers and changes of variable, Feller's operator represents locally the most general classical ordinary linear second-order differential operator $D_x^2+gD_x+h$. And for singular or discontinuous m it includes various non-classical cases previously treated singly by ad hoc methods.

The main shortcoming of A_φ is that it cannot represent all second-order classical operators globally, but only the "pure" operators $D_x^2+gD_x$ (for, in fact, if f belongs to the null space of A_φ, then $(1/\varphi)f$ must be monotone). Nonetheless, the theory of A_φ contains in a simple and elegant form the essentials of the traditional Sturm-Liouville and Weyl-Stone second-order theory. And Feller further justifies his operator by showing in the paper reviewed below that every operator A of local character preserving non-negativity at the zeroes of non-negative functions possesses an essential unique scale x and measure m such that $A=A_\varphi$ with respect to this m and x, φ being any positive function annihilated by A. In the present paper he also shows that the theory of A_φ can be reduced to that of $A_1=D_mD_x$. For the L^2 case the reduction is completely successful, for the C case only partly so.

The endpoints of I are "active", "semi-active", or "natural" according to the behavior there of functions in the null space of A_φ. This classification is a refinement of Weyl's Grenzkreis-Grenzpunkt dichotomy, and is fundamental for the whole paper. For positive eigenvalues λ it is used to describe the endpoint behavior of eigenfunctions and the common range R of the integral operators $G_\lambda^{\min}$ associated with "minimal" Green functions of A_φ, this range coinciding with the domain of A_φ precisely when the operators $A_\varphi-\lambda$ are one-one. The operator $G_\lambda^{\min}$ is a bounded, positivity-preserving right-inverse of $A_\varphi-\lambda$. For L^2 it is self-adjoint, while for C its adjoint operates on the space M of measures. Feller investigates for both L^2 and C the most general such family of right-inverses G_λ. This amounts to constructing the resolvent $(\lambda-A_\varphi|R)^{-1}$ of the restriction of A_φ to R, or, in C, to constructing the most general transition semigroup. To describe here the most general common range R one must impose conditions on the interior behavior of functions as well as on their endpoint behavior.

Misprints: P. 462, line 11 from bottom, change "$f(x_0)$" to "$f(x_0)=0$". P. 465, change second sentence of Section 2 to read "One should conceive of $m(x_1)-m(x_0)$ as the measure of the half-open interval (x_0, x_1), and ...". P. 469, replace (3.10) by (3.10a) $\int_\beta^{\beta_2} dm(x)<\infty$, in case $\beta_2<\infty$, and by (3.10b) $\int_\beta^{\beta_2} x^2dm(x)<\infty$, in case $\beta_2=\infty$. Replace the integrand in (3.11) by $|\beta_2-x|$. Delete (3.13).
H. Mirkil (Hanover, N.H.).

Feller, William. On the intrinsic form for second order differential operators. Illinois J. Math. **2** (1958), 1–18.

Let I be an open interval, and for each open subinterval J let A_J be a non-trivial linear operator with domain and range both contained in the space $C(J)$ of all continuous functions on J. Suppose also that if K is an open subinterval of J and if $g\in C(K)$ is the restriction of some f in the domain of A_J, then g belongs to the domain of A_K and A_Kg is the restriction of A_Jf (but allow the domain of A_K to be perhaps larger than the restriction of the domain of A_J). This coherent collection of A_J defines an operator A of local character. Example: An ordinary linear differential operator with continuous coefficients (and, say, leading coefficient nowhere zero). Among these classical operators the "pure" second-order ones, $aD+bD^2$, have a characteristic minimum property: if f in the domain of A has a local minimum at x, then $(Af)(x)\geq 0$. In a definitive sequence of papers [Ann. of Math. (2) **61** (1955), 90–105; Comm. Pure. Appl. Math. **8** (1955), 203–216; Proceedings

of the conference on differential equations, Univ. of Maryland, 1956, pp. 251–270; MR 16, 824, 927; 18, 575; see also the paper reviewed above], Feller has invented and studied the operator D_mD_x that naturally generalizes aD^2+bD. (The functions m and x are strictly monotone and intrinsically determined by the operator. Although m may have jumps, x is continuous and can thus serve to parametrize I.) In §§ 1–6 of the present paper, Feller proves that any operator A of local character with the minimum property (has an extension that) equals some D_mD_x, provided only that A has (on each subinterval) a 2-dimensional nullspace. And in any event he proves that A is D_mD_x on the open set of "regular" points of I, but degenerates to a first-order operator at the "singular" points. In §§ 7–9, he characterizes by means of a weaker minimum property the operator $\varphi^{-1}D_m\varphi^2D_x\varphi^{-1}$ (φ any nowhere-zero element of the null-space) that naturally generalizes aD^2+bD+c. In § 10 he provides three examples that are of great help in understanding his fivefold classification of singular points. *H. Mirkil.*

Razumihin, B. S. On the equilibrium of systems with a small multiplier. Prikl. Mat. Meh. **21** (1957), 578–580. (Russian)

Consider the system (1) $dx_i/dt=\sum p_{ij}(t)x_j$ $(i<n)$, $\mu dx_n/dt=\sum p_{nj}x_j$ together with the degenerated system (2) obtained by solving $\sum p_{nj}x_j=0$ for x_n. Then the asymptotic stability of (1) follows from that of (2), provided $\mu p_{nn}(t)<0$, μ is small and p_{nn} keeps away from zero for large t. *L. Gårding* (Lund).

Faure, Robert. Sur certaines solutions périodiques d'équations différentielles non linéaires. II. Ann. Mat. Pura Appl. (4) **43** (1957), 83–95.

The author finds conditions under which there exists a periodic solution to differential equations of the form $y''+k_1y'+k_2y=f(y, y', t)$, where k_1 and k_2 are real constants and $f(y, y', t)$ is an analytic function in y, y', t, periodic in t. Convergence questions and properties of the periodic solution are studied in detail for some particular cases. *C. J. Titus* (Ann Arbor, Mich.).

Colombo, Giuseppe. Sopra un notevole fenomeno nel campo delle vibrazioni non lineari di combinazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 726–730.

An example is given of a nonlinear differential equation of form $\ddot{q}+q-E\sin\omega t=\varepsilon f(q, \dot{q}, t, \varepsilon)$ which, for $0<\varepsilon<\varepsilon_0$, has solutions of approximate form $q=A\cos(t+\theta)+[E/(1-\omega^2)]\sin\omega t$, where A may be much larger than $E/(1-\omega^2)$. Since the nonlinear term includes terms which correspond to damping, the solutions are quite different from those of the corresponding linear equation. The proof that such solutions exist is based on a theorem of the author [Rend. Sem. Mat. Univ. Padova **27** (1957), 162–175; MR **19**, 745]. *W. Kaplan* (Ann Arbor, Mich.).

Rosenberg, R. M. On the periodic solutions of the forced oscillator equation. Quart. Appl. Math. **15** (1958), 341–354.

The differential equation $d^2x/dt^2+f(x)=P_0\cos\omega t$ is considered, where P_0 and ω are real constants. Let r be an integer and x_0 a real constant. The author shows that in order that the equation have a solution of the form $x_0\cos(\omega t/r)$, $f(x)$ must be a uniquely determined polynomial $f_r(x)$. The stability of such solutions is studied by means of a Hill equation. The theory is generalized by considering the case of small perturbations of $d^2x/dt^2+f_r(x)=P_0\cos\omega t$. *N. Levinson* (Cambridge, Mass.).

Kuškov, N. N. Qualitative investigation of a system of two differential equations. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 2(80), 195–202. (Russian)

Kuškov, N. N. Some theorems on limit cycles for systems of nonlinear oscillations. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 2(80), 203–209. (Russian)

The two papers deal with the generalized Liénard system

$$\dot{x}=y+f(x),\quad \dot{y}=g(x) \tag{1}$$

when the so called generalized Hurwitz conditions

$$xf(x),\ xg(x)<0 \text{ for } x\neq 0,\ f(0)=g(0)=0$$

fail to hold; instead one assumes that they hold outside a positive interval (x_1, x_2) but fail inside, while

$$f(0)=f(x_1)=f(x_2)=g(0)=g(x_1)=g(x_2)=0.$$

The point $(0, x_1)$ is a saddle point and basically the author gives conditions in order that the origin and $(x_2, 0)$ do or do not have "spiral-like" regions of stability. Much information about the general behavior of the paths is also given.

In the second paper under an analogous situation the author gives various (complicated) sufficiency conditions for the existence of one or two limit-cycles within certain strips. [References: Erugin, Prikl. Mat. Meh. **14** (1950), 459–512, 659–664; **16** (1952), 620–628; Malkin, ibid. **16** (1952), 365–368; Krasovskiĭ, ibid. **17** (1953), 651–672; Pliss, ibid. **17** (1953), 541–554; MR **12**, 412; **14**, 376; **14**, 48; **15**, 624, 527.] *S. Lefschetz* (Mexico, D.F.).

Chan, Chi-fen. On the uniqueness of the limit cycles of some non-linear oscillation equations. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 659–662. (Russian)

This paper deals with the general equation (generalized equation of Liénard) $\ddot{x}+f(x)\dot{x}+g(x)=0$; or, equivalently, with the system $\dot{x}=-y-F(x)$ $(F(x)=\int_0^x f(x)dx)$, $\dot{y}=g(x)$; also with the more general system $\dot{x}=-\varphi(y)-F(x)$, $\dot{y}=g(x)$. It is shown that under a variety of general assumptions there is at most one limit-cycle.

This generalizes, with added precision, results of Sansone [Univ e Politec. Torino. Rend. Sem. Mat. **10** (1951), 155–171; MR **13**, 746], Massera [Boll. Un. Mat. Ital. **9** (1954), 367–369; MR **16**, 925] and Conti [ibid. **7** (1952), 111–118; MR **14**, 558]. [Additional references: Levinson and Smith, Duke Math. J. **9** (1942), 382–403; Hudaĭ-Verenov, Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 389–396; MR **4**, 42; **19**, 653]. *S. Lefschetz.*

Mikołajska, Z. Sur un théorème de MM. N. Levinson et O. Smith relatif à l'équation différentielle des oscillations entretenues. Ann. Polon. Math. **4** (1957), 1–7.

The author presents a counterexample to a theorem of Levinson and Smith [Duke Math. J. **9** (1942), 382–403; MR **4**, 42] regarding the uniqueness of the limit cycle of the equation (*) $\ddot{x}+f(x, \dot{x})\dot{x}+g(x)=0$. The counterexample exploits certain rather singular situations, especially the possibility of $\lambda(x, v)=v^2/2+\int_0^x g(s)ds=c$ $(v=\dot{x})$ being an integral (and hence a closed solution curve in the (x, v) plane) of (*) for some constant c, which Levinson and Smith tacitly assumed in their proof not to be the case. The Levinson and Smith uniqueness criterion (their Theorem II) can apparently be appropriately

qualified so as to yield a true theorem, but the author of the present paper has not indicated what the qualifications might be. {The reviewer believes the following would suffice: suppose it is known that a closed solution curve exists in the closed region R bounded by $\lambda(x, v)=c_0>0$ and $\lambda(x, v)=c_1>c_0$. Then, if for no $c\in[c_0, c_1]$ is $\lambda(x, v)=c$ an integral of (*) and if the criterion holds for all $c\in[c_0, c_1]$ (rather than all c), the closed solution curve is unique in R. The reviewer might point out that the condition (3.1) of Levinson and Smith, from which their criterion is developed, is correct (as a sufficient condition for stability), but their proof is not.} *C. E. Langenhop.*

Ehrmann, Hans. Über die Existenz der Lösungen von Randwertaufgaben bei gewöhnlichen nichtlinearen Differentialgleichungen zweiter Ordnung. Math. Ann. **134** (1957), 167–194.

This paper is concerned with the existence of solutions to the boundary value problem (1) $y''=f(x, y, y')$, $a_1y(x_0)+a_2y'(x_0)=a_0$, $b_1y(x_1)+b_2y'(x_1)=b_0$; where $a_0^2+a_1^2\neq 0$, and $b_0^2+b_1^2\neq 0$. The function $f(x, y, v)$ is required to be defined in the region G: $x_0\leq x\leq x_1$, $-\infty<y<\infty$, $-\infty<v<\infty$ and to satisfy sufficient conditions for the existence of a continuously differentiable solution of the equation (2) $y'(x)-y'(\alpha)=\int_\alpha^x f(t, y(t), y'(t))dt$ for $x_0\leq\alpha<x_1$ which is also required to be continuous in α, $y(\alpha)$, and $y'(\alpha)$ for $x_0\leq\alpha<x_1$, $-\infty<y(\alpha)<\infty$, $-\infty<y'(\alpha)<\infty$. The author proves several existence theorems, all of which seem to follow relatively easily from the following main theorem: If, in addition to the above conditions on f, there exists a continuous function $h(y)$ defined for $-\infty<y<\infty$ and constants K_1, K_2, and K_3 such that for all $(x, y, v)\in G$ $|f(x, y, v)-h(y)|\leq K_1|y|+K_2|v|+K_3$; and if for any $C>0$ there exist positive constants ε, C, and Y such that if $x_0\leq x\leq x_1$, $|y|\geq Y$, and $|v|\leq C|y|^{1+\varepsilon}$, then $f(x, y, v)/y\leq -C$; then there exists an infinite number of solutions of (2) with $\alpha=x_0$ which satisfy the boundary conditions of (1).

R. R. Kemp (Kingston, Ont.).

Friedman, Avner. On the properties of a singular Sturm-Liouville equation determined by its spectral functions. Michigan Math. J. **4** (1957), 137–145.

The author proves two theorems on the inverse Sturm-Liouville problem considered by I. M. Gelfand and B. M. Levitan [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 309–360; MR **13**, 558; **17**, 489]. The latter proved that if $\rho(\lambda)$ is a non-decreasing function on $(-\infty, \infty)$ satisfying certain conditions, then it is the spectral function of a uniquely determined Sturm-Liouville problem $y''+(\lambda-q(x))y=0$, $y(0)\sin\alpha-y'(0)\cos\alpha=0$. Under these and some additional conditions the author obtains estimates on $q(x)$ and its derivatives and obtains a lower bound for the radius of convergence of the MacLaurin series of $q(x)$ under a further assumption. He also applies these results to a particular moment problem. *R. R. Kemp.*

Swanson, C. A. Differential operators with perturbed domains. J. Math. Mech. **6** (1957), 823–846.

The author considers eigenvalue problems for a second order ordinary linear differential operator on an interval of the type $0<x\leq b$, and perturbed problems defined on subintervals of the type $a\leq x\leq b$, where a is small and positive. Asymptotic expansions as $a\to 0$ for the eigenvalues and eigenfunctions of the perturbed problem are obtained when there is an isolated point in the spectrum of the problem on $0<x\leq b$. These expansions are of a rather general type, and result from a transition to an integral equation approach. The cases when $x=0$ is a regular singular point or an irregular singular point of finite rank are treated in some detail. {Reviewer's note: The basic operator A_0(p.826) as defined by the author is not self-adjoint as stated. It is not maximal symmetric, but its closure is self-adjoint. The perturbed operator A_a (p. 828) is not self-adjoint either, since its domain D_a is not dense in the Hilbert space $L_2(0, b)$; see condition (f) of Definition 8. However, these facts do not seem to vitiate any of his subsequent arguments, since he apparently makes no use of any of the abstract Hilbert space apparatus.} *E. A. Coddington* (Princeton, N.J.).

Kramer, Henry P. Perturbation of differential operators. Pacific J. Math. **7** (1957), 1405–1435.

The author studies the well-known theorem of Birkhoff and Tamarkin on expansion in eigenfunctions of a non-singular ordinary differential operator, and extends the abstract methods used by the present reviewer [Pacific J. Math. **4** (1954), 415–458; MR **16**, 144] to differential operators of arbitrary even order $2n$. The boundary conditions are, for reasons of simplicity, required to be of the more manageable type emphasized by Tamarkin; n boundary conditions at each end-point. It is shown that, with such boundary conditions, an operator $d^{2n}/dx^{2n}+\sum_{i=2}^{2n}Q_i d^{2n-i}/dx^{2n-i}$ possesses a set of eigenfunctions in terms of which any function in L_2 may be expanded into a series converging unconditionally in L_2; here the "coefficients" Q_i are allowed to be arbitrary bounded operators. *J. Schwartz* (Upton, N.Y.).

Villari, Gaetano. Cicli limite e fusione di separatrici. Ann. Mat. Pura Appl. (4) **42** (1956), 259–277.

This paper investigates for a generalized Liénard equation

$$\ddot{x}(t)+f(x, \lambda)\dot{x}(t)+g(x)=0, \tag{1}$$

involving a nonnegative parameter λ, the generation of limit cycles, and in general, the nature of the dependence of the solutions on λ. Solutions are studied in the phase space (x, y), $x=x(t)$, $y=\dot{x}(t)$, under the hypotheses: (a) f is continuous in $-\infty<x<\infty$, $0\leq\lambda<\infty$, is even in x, and for some α_λ (depending on λ) satisfies $f(x, \lambda)\leq 0$ for $|x|\leq\alpha_\lambda$. (b) g is an odd continuous function of x satisfying, for some β, and $x>0$, $g(x)\geq 0$ for $x\leq\beta$. (Thus the trajectories in the phase space are symmetric about the origin.) Under these hypotheses it is shown that if λ is such that $\beta\leq\alpha_\lambda$, then the system has no limit cycles, while for $\beta>\alpha_\lambda$ the system has at most one closed trajectory.

Under the additional assumptions (c) $f_x<0$, $f_\lambda>0$ for x, $\lambda>0$, subscript denoting differentiation, (d_1) $\lim_{\lambda\to\infty}\alpha_\lambda=\infty$, ($d_2$) $\lim_{\lambda\to 0+}\alpha_\lambda=0$, the author proves the following. There is a unique $\lambda^*>0$ such that: 1. For $\lambda>\lambda^*$, (1) has no periodic solutions, and the trajectories in the phase space have a certain behavior—described by a diagram. 2. For $\lambda=\lambda^*$, (1) has no periodic solutions, and some trajectories in the phase space through singular points have fused together, as shown by another diagram. 3. For $\lambda<\lambda^*$, (1) has exactly one periodic solution (and its behavior as $\lambda\to\lambda^*$ is studied).

The proof involves a rather detailed and careful analysis of the trajectories in the phase space, which cannot be described here. It is related to E. Leontovič, Dokl. Akad. Nauk SSSR (N.S.) **78** (1951), 641–644 [MR **13**, 132]. *L. Nirenberg* (New York, N.Y.).

Krasovskiĭ, N. N. The stability of quasilinear systems with after-effects. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 435–438. (Russian)

The author considers the vector equation $dx/dt = L(x(t-\theta), t)+N(x(t-\theta), t)$, where L is a linear operator, N is a non-linear operator and $h \geqq \theta \geqq 0$. Under assumptions concerning the spectrum of the operator L and the non-linear nature of N, he obtains analogues of the Poincaré-Lyapunov theorem similar to those previously derived for differential-difference equations, parabolic partial differential equations, and so on [cf. Bellman, Ann. of Math. (2) **50** (1949), 347–355, where other references may be found; MR **10**, 715].

As an example of the equations to which his results apply, he cites the system $dx_i/dt = \sum_{j=1}^{n} \int_0^h x_j(t-\theta)\,dg_{ij}(\theta)$ $(i=1, 2, \cdots, n)$. *R. Bellman.*

Swinnerton-Dyer, H. P. F. On an extremal problem. Proc. London Math. Soc. (3) **7** (1957), 568–583.

The author treats what in the parlance of control processes is usually called a "terminal control" process. Given the vector equation $\dot{x}=g(x, y)$, the problem is to determine the control vector y so as to minimize a prescribed function of the final state. If no further constraint is imposed upon y, a solution within the class of ordinary functions may not exist, and one must look for a solution within the class of distribution functions. The case of a two-dimensional system is discussed in detail, and, in addition, some comments are made concerning the corresponding problem with y constrained by conditions such as $|y_i| \leq m$.

Problems of this latter type have been treated in detail in Bellman, Glicksberg and Gross, Rend. Circ. Mat. Palermo (2) **3** (1954), 363–397 [MR **16**, 1127]; Bellman, Fleming and Widder, Ann. Mat. Pura Appl. (4) **41** (1956), 301–323 [MR **18**, 51]; and in Arrow, Karlin and Scarf, Studies in the mathematical theory of inventory and production, Stanford University Press, 1958.

R. Bellman (Santa Monica, Calif.).

See also: Functions of Complex Variables: Jurchescu. **Sequences, Series, Summability:** Siegel. **Integral Transforms:** Boxer and Thaler; Wasow. **Numerical Methods:** Proskuryakov. **Probability:** Dostupov and Pougatchev. **Mechanics of Particles and Systems:** Lewandowski. **Statistical Thermodynamics and Mechanics:** Galkin. **Control Systems:** Savinov and Citovič.

Partial Differential Equations

Chiffi, Antonio. Extensione del teorema di convergenza del metodo dei momenti di S. Faedo. Boll. Un. Mat. Ital. (3) **12** (1957), 591–595.

In 1947, S. Faedo [Ann. Scuola Norm. Sup. Pisa (3) **1** (1947), 1–41; MR **11**, 363] proved theorems of uniqueness and existence for the problem

$$-\frac{\partial^2 u}{\partial x^2}+2a_1\frac{\partial^2 u}{\partial x \partial t}+a\frac{\partial^2 u}{\partial t^2}+b_1\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial t}+cu=f,$$

where a, a_1, b, b_1, c, f are functions of (x, t) alone, in the half-strip $0 \leqq x \leqq \gamma$, $t \geqq 0$, under the boundary conditions $u(x, 0)=g(x)$, $u_t(x, 0)=g_1(x)$, $u(0, t)=f_1(t)$, $u(\gamma, t)=f_2(t)$; the only condition imposed on the coefficients was that $a>0$ in the half-strip.

In the present paper, the conditions are relaxed to $a+a_1^2>0$ in the half-strip, together with $a(x, 0)>0$ when $0 \leqq x \leqq \gamma$. *E. T. Copson* (St. Andrews).

Picone, Mauro. Riduzione dimensionale del problema dell'integrazione, in grande, dei sistemi di equazioni ai differenziali totali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 242–249.

Let $du=\sum_{i=1}^{\nu} e^{(i)}(x, u)dx$ be a system of total differential equations in n unknown functions u and ν independant variables x, the point (x, u) belonging to a given field T in $(n+\nu)$-dimensional Euclidean space, the point (x) to the field C, the orthogonal projection of T on the ν-dimensional space. $e^{(i)}$ are continuous functions in T possessing partial differential coefficients of the first order.

The system is called integrable in C, if there exists a solution u admitting continuous first partial differential coefficients, so that the point (x, u) is always in T.

The $\binom{\nu}{2}$ equations

$$\frac{\partial e^{(i)}}{\partial x_j}+\sum_{k=1}^{n} e_k^{(j)}\frac{\partial e^{(i)}}{\partial u_k}=\frac{\partial e^{(j)}}{\partial x_i}+\sum_{k=1}^{n} e_k^{(i)}\frac{\partial e^{(j)}}{\partial u_k}$$

$$(i \neq j;\ i, j=1, 2, \cdots \nu),$$

obtained by transforming the identities $\partial(\partial u/\partial x_i)/\partial x_j = \partial(\partial u/\partial x_j)/\partial x_i$ with the aid of the given system, are called the differential consequences of the system.

It is well known, that even if these equations are identically satisfied, the integrability of the system can only be stated if the field C satisfies some additional conditions, for instance, that it is starshaped. In that case, in an arbitrary given point, the solution may be constrained to coincide with an arbitrary given vector.

The formulation of such conditions encounters difficulties which increase rapidly with the number of dimensions $n+\nu$. It is therefore important to obtain a theorem, reducing the problem for a certain number of dimensions to that for a smaller number. Such a theorem is the aim of the present note. The first and most simple theorems are those for the case $n=3$, $\nu=3$, with the functions $e^{(i)}$ independant of u. They are the following.

I. A necessary condition for the integrability of the system

$$du=e(x, y, z)dx+f(x, y, z)dy+g(x, y, z)dz$$

in C is the integrability of the system

$$du=\left[e(x, y, \varphi)+g(x, y, \varphi)\frac{\partial \varphi}{\partial x}\right]dx+\left[f(x, y, \varphi)+g(x, y, \varphi)\frac{\partial \varphi}{\partial y}\right]dy$$

in A, where A is the projection of C on the x, y plane, and $z=\varphi(x, y)$ is the equation of a surface in C such that a perpendicular to a point of A on the (x, y) plane intersects the surface only in two points. If this condition is satisfied, the differential consequences of the system $e_y=f_x$, $g_x=e_z$, $g_y=f_z$ are sufficient for integrability.

II. A necessary condition for the integrability of the system

$$du=e(x, y, z)dx+f(x, y, z)dy+g(x, y, z)dz$$

in C is that the system

$$du=f(x, y, z)dy+g(x, y, z)dz$$

be always integrable in B, where B is the projection on the y, z plane of the section of C with an arbitrary plane perpendicular to the x-axis in an interval A, and that a curve $y=\varphi(x)$, $z=\psi(x)$, where φ and ψ are differentiable

functions in A, lies in C. If this condition is satisfied, it is sufficient for integrability that $f_x=e_y$, $g_x=e_z$.

The general theorem is an extension of these theorems.

H. Bremekamp.

Roşculeţ, Marcel N. Algèbres infinies, commutatives, associées à des systèmes d'équations aux dérivées partielles. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 321–371. (Romanian. Russian and French summaries)

The author has previously [same Stud. **6** (1955), 135–173; MR **17**, 145] shown how to associate, with any linear homogeneous partial differential equation with constant coefficients, an infinite algebra, associative and commutative, defined by a certain characteristic equation. The main concern in the present paper is the extension of this method to systems of linear partial differential equations, of first or of higher order, with constant coefficients, involving one or more than one unknown function. In this case, the algebra is defined by the "multiplication table" of the basis elements. To give an example, consider the following linear system, with constant coefficients, consisting of p equations in $s+2$ unknown functions of the $n+1$ independent variables $x^0, x^1, \cdots, x^n$:

$$\sum_{i=0}^{n}\left(a_{0,i}^{j}\frac{\partial V^0}{\partial x^i}+a_{1,i}^{j}\frac{\partial V^1}{\partial x^i}\right)=\sum_{i=0}^{n}\sum_{k=1}^{s}b_{k,i}^{j}\frac{\partial W^k}{\partial x^i},$$

$j=1, 2, \cdots, p$. Suppose $n-s>0$, and introduce the hypercomplex variable $\omega=\theta_0x^0+\theta_1x^1+\cdots+\theta_{n-s}x^{n-s}+\varphi_1x^{n-s+1}+\varphi_2x^{n-s+2}+\cdots+\varphi_sx^n$, where $\theta_0=1$ and the hypercomplex elements θ_i and φ_k satisfy the relations

$$\sum_{i=0}^{n}(a_{0,i}^{j}\theta_i+a_{1,i}^{j}\theta_1\theta_i)=\sum_{i=0}^{n}\sum_{k=1}^{s}b_{k,i}^{j}\varphi_k\theta_i,$$

$j=1, 2, \cdots, p$, where $\theta_{n-s+j}=\varphi_j$. If $f(\omega)$ is a monogenic function of the hypercomplex variable ω, then the $s+2$ functions $V^0=\mathrm{Re}\,f(\omega)$, $V^1=\mathrm{Re}\,[\theta_1f(\omega)]$, $W^k=\mathrm{Re}\,[\varphi_kf(\omega)]$, $k=1, 2, \cdots, s$, satisfy the given system of partial differential equations. *J. B. Diaz* (College Park, Md.).

Miranda, Carlo. Le soluzioni fondamentali delle equazioni ellittiche. Confer. Sem. Mat. Univ. Bari no. **30** (1957), 16 pp. (one plate)

Si tratta di una conferenza in cui l'A. espone in rapida e chiara sintesi le questioni relative all'esistenza e alle proprietà delle soluzioni fondamentali per le equazioni ellittiche lineari del secondo ordine

$$(1)\qquad Au\equiv\sum_{i,k=1}^{n}a_{ik}\frac{\partial^2u}{\partial x_i\partial x_k}+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu=0$$

[per una completa trattazione di dette questioni si veda Miranda, Equazioni alle derivate parziali di tipo ellittico, Springer, Berlin, 1955; MR **19**, 421]. Introdotte le cosidette funzioni di Levi o parametrix e definite le soluzioni fondamentali nel modo classico, viene tra l'altro ripreso il ben noto procedimento di E. E. Levi [Rend. Circ. Mat. Palermo **24** (1907), 275–317] per dimostrare l'esistenza di soluzioni fondamentali e, per la prima volta dopo 50 anni, viene segnalata una lacuna in detta dimostrazione e viene messa in evidenza la stretta connessione tra detta lacuna e alcuni interessanti problemi quali: 1) il teorema di unicità anche in piccolo per il problema di Cauchy relativo all'equazione omogenea aggiunta della (1); 2) una proprietà di completezza del sistema $A\alpha$ al variare della funzione α; 3) un teorema di unicità per le soluzioni dell'equazione omogenea aggiunta della (1), nulle con le loro derivate prime sulla frontiera del dominio in cui si considera la (1). Qualche breve cenno viene dato anche al caso delle equazioni di ordine superiore al secondo.

E. Magenes (Genova).

Aronszajn, N. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J. Math. Pures Appl. (9) **36** (1957), 235–249.

The chief point of the paper is the proof of the following theorem. If u is a solution of

$$|Au(x)|^2\leq M\left\{\sum_{1}^{n}\left|\frac{\partial u(x)}{\partial x^i}\right|^2+|u(x)|^2\right\},$$

where A denotes a linear elliptic differential operator of second order, in general with variable coefficients, and M a positive constant, in a domain D of the euclidean n-space, and if at some point x_0 in D u has a zero of infinitesimal order in the 1-mean, then u vanishes identically.

The following restrictions on A and u are introduced. The principal part of A, $a^{ij}\partial^2/\partial x^i\partial x^j$, should have coefficients a^{ij} of class C^2 and with second derivatives lipschitzian, all other coefficients uniformly bounded. u is in the neighbourhood of every point of D the restriction of a potential of order 2 of an L^2-function, the potential vanishing outside of a compact in E^n. That x_0 is a zero of u of infinitesimal order in the 1-mean means that

$$\int_{|x-x_0|<r}|u|dx=O(r^{\alpha+n}),\text{ for all }\alpha>0.$$

The theorem allows one to establish the uniqueness of elliptic solutions of the Cauchy problem for general linear, quasi-linear, and certain types of non-linear partial differential equations of second order. The theorem can be immediately extended to a system of m functions satisfying a system of m inequalities. *H. Bremekamp* (Delft).

Miranker, W. L. Uniqueness and representation theorems for solutions of $\Delta u+k^2u=0$ in infinite domains. J. Math. Mech. **6** (1957), 847–858.

This paper contains a number of separate but related results concerning solutions to 1) $\Delta u+k^2u=0$ in infinite three-dimensional domains D. (Similar results are claimed to hold for n dimensions.) For the case when D is the complement of a bounded domain a new proof is given for Rellich's growth estimate:

$$2)\qquad \lim_{R\to\infty}\iint_{|x|=R}|u(x)|^2dS\neq 0$$

for non-vanishing solutions to 1) in D [Jber. Deutsch. Math. Verein. **53** (1943), 57–65; MR **8**, 204]; which implies the uniqueness to solutions to 1) satisfying a radiation condition at infinity and boundary conditions on the boundary B of D. The same methods also yield the well-known representation for u as a sum of single and double layers. This result is then generalized to the case where B is unbounded. Finally a uniqueness theorem is derived for the latter case. The last two results for unbounded B seem to be new. *W. Littman* (Berkeley, Calif.).

Calabi, E. An extension of E. Hopf's maximum principle with an application to Riemannian geometry. Duke Math. J. **25** (1957), 45–56.

Soit

$$L[u]=a^{ij}(x)\frac{\partial^2u}{\partial x^i\partial x^j}+b^i(x)\frac{\partial u}{\partial x^i}\qquad(i,j=1, 2, \cdots, n)$$

un opérateur linéaire uniformément elliptique du deu-

xième ordre. On dit qu'une fonction u, semi-continue supérieurement, satisfait faiblement à $L[u] \geqq v$, si, pour chaque point x_0 et chaque $\varepsilon > 0$, il existe un voisinage V de x_0 et une fonction $u_{x_0,\varepsilon}$ de classe C^2, telle que la fonction $u - u_{x_0,\varepsilon}$ atteigne son minimum en x_0, et satisfaisant à $L[u_{x_0,\varepsilon}] \geqq v - \varepsilon$. On peut alors étendre le principe du maximum [E. Hopf, S.-B. Preuss Akad. Wiss. **1927**, 147–152] sous la forme suivante: si $u = f(x)$ est semi-continue supérieurement dans un domaine ouvert U et satisfait faiblement à $L[u] \geqq 0$ dans U, et si elle atteint un maximum local en un point x de U, elle est constante dans U.

Appliquant ces principes à la fonction $r(x)$, "distance géodésique" sur une variété riemannienne à courbure de Ricci semi-définie positive, l'A. établit l'inégalité $\Delta r \leq (n-1)/r$ au sens faible. Il en déduit que, sur une telle variété, supposée complète, il n'existe aucune solution globale de l'inéquation $\Delta u \geqq f(u)$, où $f(t)$ est une fonction positive, croissante, satisfaisant à

$$\int_0^\infty \left(\int_0^s f(t)\,dt \right)^{-1/2} ds < \infty.$$

J. Lelong (Lille).

Browder, Felix E. On some approximation methods for solutions of the Dirichlet problem for linear elliptic equations of arbitrary order. J. Math. Mech. **7** (1958), 69–80.

Let A be a linear differential operator of order $2m$, uniformly elliptic in a bounded domain $G \subset R^n$. Then there can be associated with A a Dirichlet form $A(u, v)$ (non-unique, if $m > 1$) such that $A(u, v) = (Au, v)$ holds for $u, v \in C^\infty$, one of the latter vanishing outside a compact part of G [see Browder, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 230–235, 741–747; MR **14**, 174, 473; Gårding, Math Scand. **1** (1953), 55–72; MR **16**, 366]. Moreover, suppose $A(u, u) \geqq C\|u\|$ for some $C > 0$, where the norm on the right refers to the space $L = W^{m,2}(G)$. Let P be the operator that to $u \in L$ assigns the solution of the Dirichlet problem with Dirichlet data u. Th. 1: There is a function $k \in C^\infty(G \times G)$ with $k(x, \cdot) \in L$ for $x \in G$ such that $Pu(x) = A(u, k(x, \cdot))$ for $x \in G$ and $u \in L$. Let H be the subspace of L defined by $Au = 0$. Th. 2: Let φ_j be an orthonormal basis in H; then $k(x, y) = \sum \overline{\varphi_j(x)} \varphi_j(y)$, where the convergence should be taken in the sense of L. Schwartz (uniform convergence of the function and all its derivatives on compact sets). The above theorems evidently generalize known results in the case $m = 1$ [see Bergman and Schiffer, Kernel functions and elliptic equations in mathematical physics, Academic Press, New York, 1953; MR **14**, 876]. Analogous theorems in the case when $A(u, v)$ involves only derivatives of order m. In this latter case the author gives (in Hilbert space setting) an approximation method for the Dirichlet problem which generalizes the classical Poincaré balayage process and includes also an analogue of H. A. Schwarz's alternating method. The proof makes use of a lemma (unpublished) due to Kakutani.

L. Gårding (Lund).

Narasimhan, M. S. The identity of the weak and strong extensions of a linear elliptic differential operator. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 513–514.

Let D be a partial differential operator with infinitely differentiable coefficients on an open subset Ω of R^n. The weak and strong realizations of D in Ω are defined respectively as follows. The domain of D_W consists of all f in L^2 for which Df (taken in the distribution sense) lies in L^2; $D_W f = Df$. The strong realization D_S is the closure in L^2 of D realized on all functions f in $C^\infty \cap L_2$ for which Df lies in L^2; $D_S f = Df$. Obviously $D_S \subset D_W$. If Ω is arbitrary, D formally self-adjoint and elliptic, then the author shows $D_W = D_S$. If Ω is bounded and D is uniformly elliptic, then once more $D_W = D_S$. For the case of operators of second order, similar results had been obtained previously by M. S. Birman [Dokl. Akad. Nauk SSSR (N.S.) **92** (1953), 205–208; MR **16**, 42]. *F. Browder* (New Haven, Conn.).

Narasimhan, M. S. The identity of the weak and strong extensions of a linear elliptic differential operator. II. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 620.

Extending the result of the paper reviewed above, the writer shows that if Ω is an arbitrary open set, D an elliptic partial differential operator with infinitely differentiable coefficients on Ω, then the weak and strong maximal realizations D_W and D_S of D in Ω coincide.

F. Browder (New Haven, Conn.).

Byhovskiĭ, È. B. Mixed problem for a type of partial differential equations. Vestnik Leningrad. Univ. **11** (1956), no. 19, 55–65. (Russian)

This paper considers a mixed initial-boundary value problem for a hyperbolic equation with constant coefficients in two independent variables of the form

$$\prod_{k=1}^{m} \left(\frac{\partial}{\partial x} + a_k \frac{\partial}{\partial t} \right) \prod_{k=1}^{n} \left(\frac{\partial}{\partial x} - b_k \frac{\partial}{\partial t} \right) u(x, t) = 0,$$

with $\partial^j u / \partial t^j$ given on $t = 0$ for $j \leqq m+n-1$, $\partial^k u / \partial x^k$ given on $x = 0$ for $k \leqq n-1$, and on $x = 1$ for $k \leqq m-1$. (All the a_k and b_k are positive and distinct.) If one assumes the existence of a solution of this problem having a sufficiently high number of derivatives on the closed infinite strip, the author establishes an elementary representation for such solutions. The calculations are generalized to a special case with two coincident characteristics and to more general boundary conditions with constant coefficients.

F. Browder (New Haven, Conn.).

★ **Bureau, F. J. Les solutions élémentaires et le problème de Cauchy.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 58–70. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

A brief summary is given of the concept of fundamental solution for elliptic and hyperbolic equations, and of the use of fundamental solutions for the representation of solutions of such equations. *F. Browder.*

Ladyženskaya, O. A. Solution of Cauchy's problem for hyperbolic systems by the method of finite differences. Leningrad. Gos. Univ. Uč. Zap. **144**. Ser. Mat. Nauk **23** (1952), 192–246. (Russian)

The purpose of the paper is to show the applicability of the method of finite differences for the solution of Cauchy's problem for linear and quasi-linear hyperbolic systems of partial differential equations with variable coefficients. In particular, it is shown that the replacement of $\partial u(t, x_1, \cdots, x_m)/\partial t$ by

$$(\Delta t)^{-1} [u(t+\Delta t, x_1, \cdots, x_m) - u(t, x_1, \cdots, x_m)]$$

and $\partial u(t, x_1, \cdots, x_m)/\partial x_k$ by

$$\begin{aligned}(4\Delta x)^{-1} [&u(t+\Delta t, x_1, \cdots, x_k+\Delta x, \cdots, x_m) \\ &-u(t+\Delta t, x_1, \cdots, x_k-\Delta x, \cdots, x_m) \\ &+u(t, x_1, \cdots, x_k+\Delta x, \cdots, x_m) \\ &-u(t, x_1, \cdots, x_k-\Delta x, \cdots, x_m)]\end{aligned}$$

in the hyperbolic system

$$\frac{\partial u_i}{\partial t}=\sum_{j,k=1}^{N,m} a_{ij}{}^k\frac{\partial u_j}{\partial x_k}+\sum_{j=1}^{N} b_{ij}u_j+f_i \quad (i=1, 2, \cdots, N)$$

leads to a convergent difference process as $\Delta t=c\Delta x\to 0$, where c is an arbitrary positive number. It is also shown that the replacement of $\partial u(t, x_1, \cdots, x_m)/\partial t$ by

$$(\Delta t)^{-1}\,[u(t+\Delta t, x_1, \cdots, x_m)-u(t, x_1, \cdots, x_m)]$$

and of $\partial u(t, x_1, \cdots, x_m)/\partial x_k$ by

$$(2\Delta x)^{-1}\,u(t, x_1, \cdots, x_k+\Delta x, \cdots, x_m) -u(t, x_1, \cdots, x_k-\Delta x, \cdots, x_m)]$$

leads to a convergent process as $\Delta t=c(\Delta x)^2\to 0$. For hyperbolic systems with constant coefficients, the applicability of the difference scheme employed by R. Courant, K. O. Friedrichs and H. Lewy [Math. Ann. **100** (1928), 32–74] is also shown. The corresponding Cauchy problems for the systems of Maxwell and Dirac are also solved by difference methods. *J. B. Diaz* (College Park, Md.).

Weinberger, H. F. A maximum property of Cauchy's problem. Ann. of Math. (2) **64** (1956), 505–513.

P. Germain and R. Bader [O.N.E.R.A. Publ. no. **54** (1952); Rend. Circ. Mat. Palermo (2) **2** (1953), 53–70; MR **14**, 654; **15**, 876] have shown that Tricomi's equation $yu_{xx}-u_{yy}=0$ has the following property on $y>0$: if AP and BP are its two charcteristics through P, with A and B on the singular line $y=0$, and if u vanishes on AP, then the maximum of u in the curvilinear triangle APB is attained on the segment AB of the singular line $y=0$. S. Agmon, L. Nirenberg, and M. H. Protter [Comm. Pure Appl. Math. **6** (1953), 455–470; MR **15**, 432], under suitable hypotheses on the coefficients of the equation and on the solution u, generalized this result to the general hyperbolic equations $u_{xy}+au_x+bu_y+cu=0$ and $Ku_{xx}+u_{yy}+au_x+bu_y+cu=0$. Recently, S. Bochner [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1141–1147; Proc. Confer. Differential Equations, Univ. of Maryland, 1956, pp. 23–48; MR **16**, 834; **18**, 484] has shown that the ultraspherical polynomials $P_n(x)$ have the property that if $\sum a_nP_n(x)\leq 0$ on $-1\leq x\leq 1$, then $\sum a_nP_n(x)P_n(y)\leq 0$ on $-1\leq x, y\leq 1$. Since the P_n are eigenfunctions, normalized by $P_n(1)=1$, of an ordinary differential operator L_x which is singular at $x=1$, and the second series can be considered as the solution of the hyperbolic Cauchy problem consisting of the hyperbolic partial differential equation $L_xu-L_yu=0$, together with the initial condition $u(x, 1)=\sum a_nP_n(x)$, where the initial condition on u_y is replaced by a regularity condition, Bochner's theorem can be regarded as a consequence of a maximum property of Cauchy's problem. In Section 2 of this paper the author establishes a maximum property of Cauchy's problem for the hyperbolic partial differential equation $(au_x)_x-(bu_y)_y+(cu)_x+(du)_y=0$, under suitable conditions on the coefficients. In Section 3 this maximum property is applied to ordinary differential operators and a wide generalization of Bochner's theorem is obtained. Section 4 contains a Sturm-type comparison theorem, which is applied to obtain lower bounds for eigenvalues. *J. B. Diaz.*

★ **Lieberstein, H. Melvin. On the generalized radiation problem of A. Weinstein.** Tech. Note **BN-87**, Inst. Fluid Dynamics and Appl. Math., Univ. of Maryland, 1956. iii+92 pp.

The generalized radiation problem of A. Weinstein [Summa Brasil. Math. **3** (1955), 125–147; MR **17**, 1091] consists in the determination of a twice differentiable solution of the Euler-Poisson Darboux equation

$$\text{(EPD)}\qquad L_ku\equiv u_{xx}-u_{yy}-\frac{k}{y}u_y=0$$

which satisfies the boundary conditions $u(x, 0)=f(x)$, $u(x, x)=0$. In the paper mentioned above, Weinstein formulated this radiation problem and proved the existence of a solution, under certain precise hypotheses, for the interval $-\infty<k<1$ of the parameter k. In the present paper (a doctoral dissertation written under Weinstein's direction at the University of Maryland) the author first establishes the uniqueness, under suitable hypotheses, of the solution of the generalized radiation problem. This, of course, implies the uniqueness of the solution of the "second generalized radiation problem", consisting of finding a solution of (EPD) subject to the boundary conditions $u(x, 0)=f(x)$, $u(x, x)=g(x)$. The existence of a solution to this second problem is then shown (under suitable conditions concerning the given functions $f(x)$ and $g(x)$). Finally, using a recurrence relation due to L. E. Payne but not published by him, the author solves a problem similar to the radiation problem, but with data on a non-singular line, consisting in finding a solution of (EPD) such that $u(x, b)=f(x)$ and $u(x, x)=0$ for $x>b>0$, where b is a given constant. *J. B. Diaz.*

Davis, Ruth M. On a regular Cauchy problem for the Euler-Poisson-Darboux equation. Ann. Mat. Pura Appl. (4) **42** (1956), 205–226.

Consider the Euler-Poisson-Darboux (E.P.D.) partial differential equation

$$\text{(*)}\qquad \Delta u=u_{tt}+\frac{k}{t}u_t,$$

where $u=u(x, t)=u(x_1, \cdots, x_m, t)$, the differential operator $\Delta=\partial^2/\partial x_1{}^2+\cdots+\partial^2/\partial x_m{}^2$ is the Laplacian, and k is a parameter. The singular Cauchy problem for (*) (first solved by A. Weinstein [C. R. Acad. Sci. Paris **234** (1952), 2584–2585; MR **14**, 176] and later by J. B. Diaz and H. F. Weinberger [Proc. Amer. Math. Soc. **4** (1953), 703–715; MR **15**, 321] consists in the determination of a solution of (*) meeting the following initial conditions on the "singular" plane $t=0$:

$$u(x_1, \cdots, x_m, 0)=g(x_1, \cdots, x_m),\ u_t(x_1, \cdots, x_m, 0)=0,$$

where g is a given function. In this paper (a doctoral dissertation written under Weinstein's direction at the University of Maryland) the author gives an explicit solution to the regular Cauchy problem consisting of finding a solution of (*) meeting the Cauchy conditions

$$u(x_1, \cdots, x_m, t_0)=g(x_1, \cdots, x_m),\ u_t(x_1, \cdots, x_m, 0)=0,$$

where g is a given function and $t_0>0$ (the carrier of the Cauchy data is the regular plane $t=t_0$). Her approach is based on a modification of the method of M. Riesz [Acta Math. **81** (1949), 1–223; MR **10**, 713] for solving regular Cauchy problems for self-adjoint equations. A modification is needed in the method of Riesz because (*) is not self-adjoint. The main interest lies in the fact that she has determined explicitly, for the first time, the "elementary solution in the sense of M. Riesz", V^α, for the non-trivial non-self-adjoint equation (*), and carried out the requisite analytic continuation in order to obtain an explicit formula for the regular Cauchy problem in question.

J. B. Diaz (College Park, Md.).

Walter, Wolfgang. Über die Euler-Poisson-Darboux-Gleichung. Math. Z. **67** (1957), 361–376.

This paper contains several new results on the initial value problem

$$u_{rr}+\frac{k}{r}u_r-\Delta u=0,\ r>0;$$

$$\lim_{r\to 0} u(r, x_1, \cdots, x_n)=f(x_1, \cdots, x_n);$$

$$\lim_{r\to 0}\frac{\partial u}{\partial r}(r, x_1, \cdots, x_n)=0;$$

where u is a real function of $r, x_1, \cdots, x_n$, k is a constant, and Δ is the Laplace operator with respect to the variables x_i. In the case when f is analytic in all its variables, and $k\neq -1, -3, \cdots$, the solution u is represented as a power series in r^2, with the coefficient of $r^{2\nu}$ proportional to $\Delta^\nu f$. This is a direct generalization of a result of Hyman [Nederl. Akad. Wetensch. Proc. Ser. A. **57** (1954), 408–413; MR **16**, 368]. In the cases $k=-1, -3, \cdots$, the solution u is represented as a similar series containing terms in $\log r$.

It is also shown that the Weinstein mean value representation gives a solution of the initial value problem under rather weak conditions. In fact, for $k\geqq n+3$, it is sufficient to assume that f is continuous. For $k\geqq n+5$, the initial values are taken on for almost all x if f is only locally summable. On the other hand, if f is continuously differentiable and $k\geqq n+1$, then $\partial u/\partial x_i\to\partial f/\partial x_i$ as $r\to 0$. A similar result holds for the second derivatives if f is twice continuously differentiable and $k\geqq n-1$.

It is pointed out that the known uniqueness theorems for $k\geqq 0$ [Ásgeirsson, Math. Ann. **113** (1936), 321–346; Courant and Hilbert, Methoden der mathematischen Physik, Bd. 2, Springer, Berlin, 1937, S. 381–382] rely implicitly on the additional assumption $\partial u/\partial x_i\to\partial f/\partial x_i$. (Actually, this need only hold in a mean square sense.) The author shows that if this additional assumption is made, the condition on $\partial u/\partial r$ may be replaced by the weaker condition $\lim_{r\to 0} r^k(\partial u/\partial r)=0$.

H. F. Weinberger (College Park, Md.).

Ficken, F. A.; and Fleishman, B. A. Initial value problems and time-periodic solutions for a nonlinear wave equation. Comm. Pure Appl. Math. **10** (1957), 331–356.

Existence and uniqueness theorems for various initial value problems are established for the nonlinear one-dimensional wave equation

$$(*)\qquad u_{xx}-u_{tt}-2\alpha u_t-\alpha u=\varepsilon u^3+b.$$

This equation governs the transverse displacement of a string with resistance proportional to the velocity and with nonlinear constraints. It may also be interpreted as a nonlinear telegraph equation.

The technique consists of the reduction of (*) to a nonlinear integral equation. This integral equation is then solved by Banach space methods. Estimates are obtained for the derivatives.

A ρ-preferred solution of (*) is one uniformly bounded by a constant ρ depending on the parameters in the differential equation. It is shown that the difference between two ρ-preferred solutions tends to zero as $t\to\infty$. The main result states that if b is periodic in t and if (*) has a ρ-preferred solution, then it has a unique ρ-preferred solution with the same period as b. The comprehensive results in this paper include various partial results announced previously by the authors. *M. H. Protter.*

Peyser, Gideon. Energy integrals for the mixed problem in hyperbolic partial differential equations of higher order. J. Math. Mech. **6** (1957), 641–653.

This paper solves a very general class of mixed initial-boundary value problems for a single hyperbolic equation $Eu=f$ of arbitrary order $m+1$ in two independent variables x, y. The initial line is the positive x axis where Cauchy data are prescribed, the boundary is the positive y axis where a certain number j_0 of linear combinations $\Lambda_i u$ of derivatives of u of order m (plus lower order derivatives) are given; j_0 is equal to the number of characteristics entering the positive quadrant through the boundary, and Λ_i are required to be linearly independent (mod. terms of order less than m) of the characteristic operators P_j corresponding to the incoming characteristics. P_j is defined as $\prod_{k\neq j}\partial_k$, where $\partial_k=\partial/\partial t+\lambda_k\partial/\partial x$ is differentiation in the kth characteristic direction of E; i.e., $E=\prod\partial_k$ (mod. terms of order $\leqq m$). The author introduces the auxiliary operator F of order m, defined as $F=\sum_{i\leq j_0}P_i+c\sum_{j_0+1}P_i$, c some positive constant, and forms the energy integral over the half-slab $y\leqq y_0$ in the positive quadrant:

$$\iint FuEu=\int_0^\infty Q(u)dx\Big|_{y=0}^{y=y_0}+\int_0^{y_0}R(u)dy|_{x=0}+\iint S(u),$$

where Q and R are quadratic forms in the mth partials, S in the partials of order $\leqq m$. It follows from the known theory for the pure initial value problem that Q is positive definite. The author observes — and this is his decisive contribution — that R too can be made positive definite for all functions satisfying the homogeneous boundary conditions $\Lambda_i u=0$, provided the constant c is chosen large enough.

From this energy identity an inequality can be easily derived, which is used by the author to prove the convergence of an iteration scheme based on integration along characteristics.

This problem has been treated previously by Campbell and Robinson [Proc. London Math. Soc. (3) **5** (1955), 129–147; MR **16**, 1116] using estimates in the maximum norm. Such a method is restricted to the case of two independent variables, while there is hope that the method of the present paper can be extended. How far it will carry is not clear at present.

A special case of this problem has been treated previously by energy integrals by Thomée [Math. Scand. **3** (1955), 115–123; MR **17**, 270]. Independently and at about the same time, he has extended his work to the general case [see article reviewed below]. *P. D. Lax.*

Thomée, Vidar. Estimates of the Friedrichs-Lewy type for mixed problems in the theory of linear hyperbolic differential equations in two independent variables. Math. Scand. **5** (1957), 93–113.

Let $u(x, t)=(u_1, u_2, \cdots, u_n)$ be a real vector function of class C^1 in a region V of the x, t-plane. The hyperbolic operator $L=(L_1, \cdots, L_n)$ is defined by

$$L_i u=(D_t-\alpha_i D_x)u_i+\sum_{k=1}^n a_{ik}u_k\ (i=1, 2, \cdots, n),$$

where $a_{ik}(x, t)$ are C^0, $\alpha_i(x, t)$ are C^1, and $\alpha_1\leqq\alpha_2\leqq\cdots\leqq\alpha_n$. The characteristic form of L,

$$P=\prod_{i=1}^n(\tau-\alpha_i\xi),$$

divides the ξ, τ-plane into parts Σ_i ($i=0, \cdots, n$), where Σ_i is the set of points at which exactly i factors of P are negative. Correspondingly, the boundary S of V is

divided into parts S_i, so that a boundary point is in S_i when the exterior normal (ν_ξ, ν_τ) is in Σ_i. Certain assumptions are made regarding the arrangement of these segments S_i of the boundary.

On each segment S_i $(i\neq 0)$ an n-component linear boundary operator $l_i=(l_{i1}, l_{i2}, \cdots, l_{in})$ is defined by

$$l_{ik}(u)=\sum_{j=1}^{n} l_{ik}{}^{j}u_j,$$

with coefficients which are continuous on S_i and satisfy certain linear relations.

With $R=V, S$, or S_i, the norm of an m-component function on R is given by

$$\|\varphi\|_R{}^2=\int_R \sum_{j=1}^{m} \varphi_j{}^2 dR.$$

For the product operator

$$\mathfrak{L}u=(Lu, l_1u, l_2u, \cdots, l_nu)$$

we have

$$\|\mathfrak{L}u\|_{V,S}{}^2=\|Lu\|_V{}^2+\sum_{i=1}^{n}\|l_iu\|_{S_i}{}^2.$$

The main theorem of part I of the paper under review may now be stated: There exists a constant C such that, independent of u,

$$\|u\|_V\leq C\|\mathfrak{L}u\|_{V,S} \text{ and } \|u\|_S\leq C\|\mathfrak{L}u\|_{V,S}.$$

As a corollary: The boundary problem $\mathfrak{L}u=0$ is "well set" in the sense of Hadamard. The proof of the theorem employs "energy integral" estimates.

In part II of the paper, a single-component nth order hyperbolic operator is dealt with by referring to the preceding case. The conclusions reached in part II are parallel to those of part I. *R. McKelvey* (Boulder, Colo.).

Lasota, A. Sur un nouveau problème aux limites relatif à l'équation de la corde vibrante. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 843–846, LXXIII. (Russian summary)

Let $f(x, y, u, p, q)$, $g(x, u, q)$, $h(y, u, p)$ be continuous, for (x, y) on R: $0\leq x\leq a$, $0\leq y\leq b$, and for all (u, p, q), and satisfy $|f|$, $|g|$, $|h|\leq M$ and uniform Lipschitz conditions with respect to p, q, with Lipschitz constants F, G, H, respectively. Let $k_i(x, u)$, $l_i(y, u)$, where $i=1, 2, 3$, be continuous for $(x, y)\in R$ and all u, and satisfy $0\leq k_i\leq a$, $0\leq l_i\leq b$ and uniform Lipschitz conditions with respect to u, with a common Lipschitz constant K. If a, b, F, G, H, K, M satisfy certain inequalities (for example, if a, b, MK, GH are sufficiently small), then there exists a function $u=u(x, y)$ on R satisfying $u(x_0, y_0)=u_0$ and possessing continuous derivatives u_x, u_y and u_{xy} such that $u_{xy}\equiv f(x, y, u(x, y), u_x(x, y), u_y(x, y))$; the surfaces $u=u(x, y)$, $y=k_i(x, u)$ intersect along an arc $y=y_1(x)$, and $u_x(x, y_1(x))\equiv g(x, u(x, y_2(x)))$, $u_y(x, y_3(x))$ for $0\leq x\leq a$; finally, the surfaces $u=u(x, y)$, $x=l_i(y, u)$ intersect along an arc $x=x_1(y)$, and $u_y(x_1(y), y)\equiv h(y, u(x_2(y), y), u_x(x_3(y), y)$ for $0\leq y\leq b$. The proof depends on Schauder's fixed point theorem [for references to related results cf. Szmydt, same Bull. **5** (1957), 571–575, XLIX; MR **19**, 748].

P. Hartman (Los Angeles, Calif.).

Šantavý, Ivan. Der Reziprozitätssatz der nichtharmonischen Wellenvorgänge. Apl. Mat. **2** (1957), 390–397. (Czech. Russian and German summaries)

The author considers a function $U=U(x, y, z, t)$ satisfying the wave equation

$$\Delta^2U=\frac{1}{a^2}\frac{\partial^2U}{\partial t^2}$$

inside an open set O which is bounded by a closed surface of the Liapunov type, with U vanishing at this boundary. At a point $A(x_A, y_A, z_A)$, $A\in O$, is a source. The author constructs a function satisfying the wave equation at every point of O except at the source A:

$$\frac{1}{r_A}F\left(t-\frac{r_A}{a}\right),$$

where r_A is the distance between points $P(x, y, z)$ and $A(x_A, y_A, z_A)$ and the function F must satisfy a number of conditions of continuity. The solution of the whole boundary value problem is

$$U_A=\frac{1}{r_A}F\left(t-\frac{r_A}{a}\right)+U_{OA}(x, y, z, t),$$

where U_{OA} is defined everywhere in O (including the source point A) and chosen so that U_A satisfies both the wave equations and the boundary condition $U=0$. The subscript A in U_A means that U is caused by a source at the point A, and consequently U_B would mean that U is caused by a source at the point B. Using the Laplace transformation the author proves that for the function U is valid the reciprocity relation $U_A(B, t)=U_B(A, t)$.

T. Leser (Aberdeen, Md.).

Ventcel', A. D. Semigroups of operators that correspond to a generalized differential operator of second order. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 269–272. (Russian)

The author corrects an error in Feller, Ann. of Math. (2) **55** (1952), 468–519 [MR **13**, 948], by showing that the lateral conditions assumed for the diffusion operator in the main theorem of that paper are not the most general possible. Feller himself has recently published another paper, Illinois J. Math. **1** (1957), 459–504, [MR **19**, 1052] in which this theorem is corrected and greatly generalized. The notational confusion leading to the error in the Annals paper is explained in footnotes 10 and 20 of the Illinois Journal paper. *H. Mirkil* (Hanover, N.H.).

Douglas, Jim, Jr. A uniqueness theorem for the solution of a Stefan problem. Proc. Amer. Math. Soc. **8** (1957), 402–408.

The author proves the uniqueness of the solution $u(x, t)$ of the following problem involving a free boundary: $u(x, t)$ is to be defined in A: $0<x<x(t)$, $t>0$, with $x(t)$ also to be determined, such that $u_{xx}=f(u)u_t$ in A; $u_x(0, t)=-a$, $u(x(t), t)=0$, $dx/dt=b-cu_x(x(t), t)$ for $t>0$ and $x(0)=0$. Here a, b, c are nonnegative constants, $f(u)>0$ and $f'(u)\geq 0$. The proof makes use of some results by Douglas and Gallie [Duke Math. J. **22** (1955), 557–571; MR **17**, 1241], and also makes use of the fact that solutions can be approximated by solutions of corresponding finite difference schemes. (The use of finite difference schemes can, however, be avoided.) Analyticity of $x(t)$ is not assumed. *L. Nirenberg* (New York, N.Y.).

Pini, Bruno. Sul primo problema di valori al contorno per l'equazione parabolica non lineare del secondo ordine. Rend. Sem. Mat. Univ. Padova **27** (1957), 149–161.

Sia R il rettangolo: $0\leq x\leq 1$, $0\leq y\leq h$ ed S la frontiera di R privata dei punti $0<x<1$, $y=0$. Il problema considerato è il seguente: (A) $\mathfrak{L}[u]=f(x, y, u, u_x)$ per

$(x, y) \in R-S$; $u=u_0$ per $(x, y) \in S$. Una funzione $u(x, y)$ è detta "regolare" se: 1) u è continua in R e u_x è continua in $R-S$, 2) esiste per ogni $P \equiv (x, y) \in R-S$ l'operatore $\mathfrak{M}[u]$ in un senso conveniente come introdotto dall'A. in un precedente lavoro [Boll. Un. Mat. Ital. (3) 9 (1954), 244–250; MR 16, 698]. In particolare, se la funzione u soddisfa la 2) ed inoltre ammette le derivate u_y e u_{xx}, si ha: $\mathfrak{M}[u]=u_{xx}(P)+u_y(P)$; pertanto il problema (A) è di tipo parabolico generalizzato.

Supponiamo, per semplicità, $f(P, z, p)$ definita per $P \equiv (x, y) \in R$, $-\infty<z, p<+\infty$. Il problema (A) ammette al più una soluzione regolare se $f(P, z, p)$ soddisfa una delle condizioni seguenti: I) $f(P, z, p) \leqq f(P, \bar{z}, p)$ per $z<\bar{z}$; II) $|f(P, z, p)-f(P, \bar{z}, p)| \leqq K|z-\bar{z}|$ (K=costante).

Detto (A_0) il problema (A) quando $u_0=0$, l'esistenza della soluzione regolare di (A_0) è, ad esempio, assicurata se: III) $f(P, 0, p)$ è limitata e $|f(P, z, p)-f(P, \bar{z}, p)| < H|z-\bar{z}|^\lambda$ ($H, \lambda<1$, costanti positive). La condizione III) e la non decrescenza di $f(P, z, p)$ rispetto a z (cond. I)) implicano dunque l'esistenza e l'unicità della soluzione regolare e ciò contiene l'affermazione del teorema 10 di pag. 161. *G. Stampacchia* (Genova).

Ghizzetti, Aldo. Sugli integrali doppi di espressioni lineari alle derivate parziali. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 276–281.

Ghizzetti, Aldo. Sugli integrali doppi di espressioni lineari alle derivate parziali. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 430–433.

Sono indicati: con $\mathscr{E}$ un operatore differenziale lineare alle derivate parziali, di ordine $n \geqq 2$, in due variabili indipendenti x, y; con T un dominio regolare del piano x,y; con $u=u(x,y)$ una funzione di classe C^n. L'Autore enuncia il problema di cercare le condizioni necessarie e sufficienti (per T e per $\mathscr{E}$) sotto cui l'integrale

$$\Phi(u)=\iint_T \mathscr{E}(u)dxdy$$

è un funzionale puntuale, cioè dipendente soltanto dai valori che u e le derivate parziali della u, fino all'ordine $n-2$, assumono nei punti angolosi della frontiera $\mathscr{F}T$ di T. Sotto opportune ipotesi qualitative, per i coefficienti che figurano in $\mathscr{E}$ e per la $\mathscr{F}T$, l'Autore risolve analiticamente questo problema nei casi $n=2$, $n=3$. Le formule da lui trovate vengono poi opportunamente interpretate, pervenendo ai seguenti enunciati.

Anzitutto, affinchè $\Phi(u)$ sia puntuale, è necessario che $\mathscr{F}T$ sia costituita da curve caratteristiche dell'operatore $\mathscr{E}$. In secondo luogo, richiedendo che le condizioni cercate valgano non soltanto per un particolare dominio T limitato da caratteristiche, ma per tutti i dominii T limitati da caratteristiche o almeno per una parte di essi, l'Autore, dopo aver introdotto un'opportuna semplificazione, distingue tre casi e, in ciascuno di essi, deduce tutte e sole le soluzioni cercate, da quelle relative a particolari esempi tipici elementari.

Caso I) L'operatore $\mathscr{E}$ è di 2° ordine e di tipo iperbolico. È richiesto che $\Phi(u)$ sia puntuale per ogni dominio T limitato da curve appartenenti ai due sistemi di caratteristiche. L'esempio tipico è:

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{\partial^2 u}{\partial x \partial y}\,dxdy=u(x_0, y_0)-u(x_1, y_0)+u(x_1, y_1)-u(x_0, y_1).$$

Caso II) $\mathscr{E}$ è di 3° ordine e da ogni punto escono due direzioni caratteristiche reali e distinte, l'una doppia e l'altra semplice. È richiesto che $\Phi(u)$ sia puntuale per ogni dominio T limitato da curve appartenenti ai due sistemi di caratteristiche. Viene, in tal caso, introdotto un certo operatore lineare

$$\mathscr{G}=\alpha(x, y)\frac{\partial}{\partial x}+\beta(x, y)\frac{\partial}{\partial y}+\gamma(x, y)$$

e l'esempio tipico è:

$$(1)\quad \int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{\partial^2 \mathscr{G}(u)}{\partial x \partial y}\,dxdy=[\mathscr{G}(u)]_{(x_0,y_0)}-[\mathscr{G}(u)]_{(x_1,y_0)}+[\mathscr{G}(u)]_{(x_1,y_1)}-[\mathscr{G}(u)]_{(x_0,y_1)},$$

con $\alpha=0$, $\beta \neq 0$.

Caso III) $\mathscr{E}$ è di 3° ordine e da ogni punto escono tre direzioni caratteristiche reali e distinte. È richiesto che $\Phi(u)$ sia puntuale per ogni dominio T limitato a) da curve appartenenti a due prefissati dei tre sistemi di caratteristiche, oppure b) da curve appartenenti a tutti e tre i sistemi di caratteristiche. Per il sottocaso a) l'esempio tipico è dato ancora dalla formula (1), ma con $\alpha \neq 0$, $\beta \neq 0$. Per il sottocaso b) l'esempio tipico è:

$$\iint_T \left(\frac{\partial^3 u}{\partial x^2 \partial y}-\frac{\partial^3 u}{\partial x \partial y^2}\right)dxdy=[u_x-u_y]_{(x_0,y_0)}+[u_y]_{(c-y_0,y_0)}-[u_x]_{(x_0,c-x_0)},$$

ove T è il triangolo di vertici (x_0, y_0), $(c-y_0, y_0)$, $(x_0, c-x_0)$.

La deduzione dagli esempi tipici è fatta eseguendo opportuni cambiamenti sia sulle variabili indipendenti, sia sulla funzione in esame. *T. Viola* (Roma).

Berezanskiĭ, Yu. M. Generalization of Bochner's theorem to expansions according to eigenfunctions of partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 893–896. (Russian)

Let L be a partial differential operator with smooth coefficients on an open set G of R^n, L' its formal adjoint, $\bar{L}$ the differential operator whose coefficients are the complex conjugates of those of L. The continuous positive-definite kernel $\psi(x, y, \lambda)$ defined for x and y in G and real λ is said to be an elementary kernel for L if ψ is r-times continuously differentiable in each variable and, for every λ, $L_x\psi(x, y, \lambda)=\lambda\psi(x, y, \lambda)$, $\bar{L}_y\psi(x, y, \lambda)=\lambda\psi(x, y, \lambda)$. The author sketches a proof for the following theorem: In order that a continuous positive-definite kernel $K(x, y)$ $(x, y \in G)$ may be written in the form $K(x, y)=\int \psi(x, y, \lambda)d\rho(\lambda)$, where $\psi(x, y, \lambda)$ is an elementary kernel for L and $\rho(\lambda)$ is a non-decreasing function, it is necessary and sufficient that the operator L' defined on the infinitely differentiable functions with compact support in G be symmetric relative to the inner product $\langle f, g\rangle=\int_{G\times G} K(x, y)f(y)\bar{g}(x)\,dxdy$. If $K(x, y)$ is r-times differentiable with respect to each variable, the symmetry condition becomes $L_xK(x, y)=\bar{L}_yK(x, y)$. From this theorem, the author derives Bochner's theorem on the representation of positive-definite functions as Fourier-Stieltjes transforms of positive measures, a generalization given by M. Krein to eigenfunctions of ordinary differential operators, and S. Bernstein's theorem on absolutely monotonic functions.

F. Browder (New Haven, Conn.).

Browder, Felix E. Eigenfunction expansions for formally self-adjoint partial differential operators. I, II. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 769–771, 870–872.

Consider the pair (A, B), where each component is a linear, formally self-adjoint, differential operator in a

domain $G \subset R^n$. Suppose that B is positive (i.e., $(Bf, f) > 0$ for $0 \neq f \in C_0^\infty(G)$). Gelfand and Kostyuchenko [Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 349–352; MR **17**, 388] have established the existence of a complete set of (generalized) eigenfunctions of $A - \lambda B$ under the auxiliary hypothesis that there exists a hyper-maximal realization of $B^{-1}A$ in the Hilbert space, obtained by completion of $C_0^\infty(G)$ in the norm (Bf, f). In the first of the two notes the author obtains an eigenfunction expansion corresponding to each generalized spectral resolution in the sense of Neumark. The proof depends on a theorem of Gelfand [Mat. Sb. N.S. **4**(**46**) (1938), 235–284]. In the second note the author studies the case when (A, B) satisfies a certain ellipticity condition, now relying on the Radon-Nikodym theorem. In this case the proof can be carried through under less assumptions on the coefficients of the operators involved. *L. Gårding* (Lund).

See also: Harmonic Functions, Convex Functions: Boggio. **Ordinary Differential Equations:** Feller. **Numerical Methods:** Diaz; Koval'; Conte. **Mechanics of Particles and Systems:** Duffin and Schild. **Elasticity, Plasticity:** Nicolovius; Payne and Weinberger; Petrova. **Classical Thermodynamics, Heat Transfer:** Chow; Jung.

Difference Equations, Functional Equations

Nagell, Trygve. On linear recurrences with constant coefficients. Ark. Mat. **3** (1958), 395–401.

Let $\{A_n\}$ ($n = 0, 1, \cdots$) be a sequence satisfying a linear difference equation with constant coefficients which is homogeneous except for at most a constant. It is shown, without direct application of the theory of difference equations, that there exists a unique rational function $f(z)$ such that $f(z) = \sum_{n=0}^{\infty} A_n z^n$ within some circle of convergence. If this generating function is known, it is shown how an expression for the general term of the sequence can be obtained. {Reviewer's note: For the case of a homogeneous linear difference equation with constant coefficients these results are well known [see, e.g., G. Chrystal, Algebra, an elementary text-book, vol. 2, 6th ed., Chelsea, New York, 1952, pp. 411–415], and the extension to the case of a difference equation homogeneous except for a polynomial in n is rather trivial, since any solution of such an equation satisfies a homogeneous equation of order greater by the degree of the polynomial plus one.} *T. N. E. Greville* (Washington, D.C.).

Boas, R. P., Jr. On generalized averaging operators. Canad. J. Math. **10** (1958), 122–126.

Let $\nabla_h f(z) = \frac{1}{2}[f(z+h) + f(z)]$. For real λ and h, D. B. Sumner [same J. **8** (1956), 437–446; MR **19**, 151] defined $\nabla_h^\lambda f(z)$ by means of a definite integral, valid for entire functions $f(z)$ of exponential type $< \pi/|h|$. Boas now obtains an equivalent definition as a particular case of a more general situation. Let $f(z)$ be of finite exponential type, S its conjugate indicator diagram, and $F(w)$ the Borel-Laplace transform of $f(z)$. If $\phi(w)$ is regular on S, and C is a contour surrounding S and sufficiently close to S, then

$$(*) \qquad [\phi(D)]^\lambda f = (2\pi i)^{-1} \int_C F(w)[\phi(w)]^\lambda e^{zw} dw,$$

valid for all positive integers λ in any case, and for arbitrary real and complex numbers λ if $\phi(w)$ has no zeros on S. (*) reduces to the Sumner operator $\phi(D) = \nabla_h$ for the choice $\phi(w) = \frac{1}{2}(1 + e^{hw})$; and the equivalence of the two definitions is established. Boas also gives a discussion, supplementing that of Sumner, on solutions of the equation $\nabla_h^\lambda f(z) = 0$. *I. M. Sheffer* (State College, Pa.).

See also: Ordinary Differential Equations: Bellert.

Integral and Integrodifferential Equations

Schlesinger, Stewart. Approximating eigenvalues and eigenfunctions of symmetric kernels. J. Soc. Indust. Appl. Math. **5** (1957), 1–14.

Most of the results of this paper are essentially contained in Courant and Hilbert, Methoden der mathematischen Physik, Bd. 1 [2. Aufl., Springer, Berlin, 1931, K. 3]. Section 4 applies to these results theorems of the following kind: If K is a bounded linear transformation from L_2 to C and if the sequence $\varphi^{(N)}$ converges strongly to φ in L_2, then $K\varphi^{(N)}$ converges strongly to $K\varphi$ in C. *H. F. Weinberger* (College Park, Md.).

Kohn, J. J. Singular integral equations for differential forms on Riemannian manifolds. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 650–653.

L'A. étudie l'opérateur singulier

$$J\varphi(x) = \xi(x) \cdot \varphi(x) + \int_M G(x, y) \Lambda^* \varphi(y)$$

opérant sur les formes différentielles φ d'une variété riemannienne M de dimension m; $G(x, y)$ est une forme différentielle double telle que $r^m(x, y)G(x, y)$ ait une limite radiale $F_x(u)$ quand $y \to x$ suivant une direction quelconque u, cette convergence étant précisée par des conditions supplémentaires. L'équation $J\varphi = \alpha$ est dite régularisable s'il existe un opérateur K tel que $K \circ J\varphi = K\alpha$ soit une équation de Fredholm ayant les mêmes solutions. L'A. construit un opérateur S_M tel que $J\varphi = \alpha$ est régularisable si et seulement si $S_M J$ a un inverse. *J. Leleng* (Zbl **71** (1958), 323).

See also: Harmonic Functions, Convex Functions: Schiffer.

Calculus of Variations

See: Functions of Real Variables: Baiada. **Ordinary Differential Equations:** Swinnerton-Dyer. **Mechanics of Particles and Systems:** Toraldo di Francia. **Quantum Mechanics:** Kikuta.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Maharadze, L. M. Locally nilpotent ideals in topological rings. Mat. Sb. N.S. **41**(**83**) (1957), 395–414. (Russian)

Consider the following properties of a topological ring S: I. S admits a complete system of neighborhoods $\{A_\alpha\}$ of 0 that are closed right ideals; II. S has a bounded neighborhood of 0, i.e., an A_α such that for every A_β there is an A_γ such that $A_\alpha A_\gamma \subset A_\beta$. S is said to be nilpotent,

following Kaplansky, if for every A_α there is a positive integer n such that $[S^n]\subset A_\alpha$. A subset B of S is said to be locally nilpotent if for every finite $F\subset B$ the closed subring of S generated by F is nilpotent. A third condition that may be imposed on S is: III. S has a locally nilpotent neighborhood of 0. The locally nilpotent radical $N(S)$ of S is defined as the sum of all locally nilpotent closed 2-sided ideals of S. Th. 1: If S satisfies conditions I and II, then $N(S)$ is a locally nilpotent, possibly non-closed, 2-sided ideal in S. Th. 2: If S satisfies conditions I, II, and III, then $N(S)$ contains all locally nilpotent closed right (or left) ideals of S. Th. 3. If S satisfies conditions I, II, and III, then the locally nilpotent radical of $S/N(S)$ is $\{0\}$. Th. 4: If S satisfies condition III, then $N(S)$ is open. Several other theorems of the same genre are proved. [For earlier studies, on which the present one is based, see Kaplansky, Amer. J. Math. **74** (1952), 929–935 [MR **14**, 348] and Levitzki, Bull. Amer. Math. Soc. **52** (1946), 1033–1035 [MR **8**, 435], as well as the literature mentioned in the reviews just cited.] *E. Hewitt.*

Maharadze, L. Topological nilpotent rings with minimal condition. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 181–186. (Russian).

Notation and terminology are as in the preceding review. The topological ring S is said to satisfy the minimal condition for closed subgroups (closed ideals) if for every descending sequence $H_1\supset H_2\supset\cdots\supset H_n\supset\cdots$ of closed subgroups (closed ideals) and every A_α there is a positive integer $n(\alpha)$ such that $H_{n(\alpha)}\subset A_\alpha$. Th. 1: Let S have a complete system of neighborhoods of 0 that are closed additive subgroups. If S is nilpotent and satisfies the minimal condition for 2-sided closed ideals, then S also satisfies the minimal condition for closed subgroups. Th. 2: Let S be locally compact, totally disconnected, and satisfy the minimal condition for 2-sided closed ideals. Let H be any compact open subring of S. Then S is the local direct sum of its primary subrings S_p ($p=2, 3, 5, \cdots$) with distinguished subrings $S_p\cap H$, which are the primary components of H [for a description of this construction see Vilenkin, Uspehi Mat. Nauk (N.S.) **5** (1950), no. 4(38), 19–74; MR **12**, 78]. *E. Hewitt* (Seattle, Wash.).

Topological Groups

Schwarz, Štefan. The theory of characters of commutative Hausdorff bicompact semigroups. Czechoslovak Math. J. **6** (**81**) (1956),330–364. (Russian summary)

This paper extends previous studies by the author [same J. **4**(**79**) (1954), 219–247, 291–295, 296–313; MR **16**, 1085, 1086]. Let S be a compact commutative semigroup, and $a\in S$. The set $\{a, a^2, a^3, \cdots\}^-$ contains exactly one idempotent e, and a is said to belong to e. The set of all $x\in S$ that belong to a given idempotent $e\in S$ forms a subsemigroup of S, denoted P_e. There is a largest subgroup of S containing e, denoted G_e. After some preliminaries, of independent interest, about ideals and prime ideals (i.e., whose complements are subsemigroups) in S, the author proves the following main theorem. Let χ be a continuous complex function on S such that $\chi(a)\chi(b)=\chi(ab)$. Let $\mathfrak{p}=\{x:x\in S, \chi(x)=0\}$, $J=\{x:x\in S, |\chi(x)|<1\}$, $Q=\{x:x\in S, |\chi(x)|=1\}$. Then J is an open prime ideal in S, and $Q=S\cap J'$ is a closed subsemigroup of S. Q is the union of all of the subsemigroups P_e for which $\chi(e)=1$ and J is the union of all P_e for which $\chi(e)=0$. $\mathfrak{p}$ is a closed prime ideal in S containing all maximal groups G_e such that $e\in J$. Many examples are given. Ideals in S of the form $\{x:x\in S, \chi(x)=0\}$ for χ as above are studied and more or less identified. [For other studies of functions χ, see Hewitt and Zuckerman, Acta Math. **93** (1955), 67–119; Trans. Amer. Math. Soc. **83** (1956), 70–97; MR **17**, 1048; **18**, 465.] *E. Hewitt* (Seattle, Wash.).

Hulanicki, A. On locally compact topological groups of power of continuum. Fund. Math. **44** (1957), 156–158.

The author shows that, on the assumption of the continuum hypothesis, a locally compact topological group of the power of the continuum must be metrizable. *K. de Leeuw* (Stanford, Calif.).

Hulanicki, A. Algebraic characterisation of abelian groups which admits compact topologies. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 405–406.

In discussing abelian groups which admit a compact topology, the reviewer [Infinite abelian groups, Univ. of Michigan Press, 1954; MR **16**, 444] left open a cardinal number question. The author announces the full result; the proof is to appear in Fund. Math. [cf. the review below]. *I. Kaplansky* (Princeton, N. J.).

Hulanicki, A. Algebraic characterization of abelian divisible groups which admit compact topologies. Fund. Math. **44** (1957), 192–197.

Let R be the additive group of rationals and let $Z(p^\infty)$ be the p-primary component of the rationals mod 1 (p=prime). A divisible abelian group G is then the direct sum of a number of copies of R and $Z(p^\infty)$, $p=2, 3, \cdots$. Let $c(R)$ denote the cardinal number of the direct summands which are equal to R and let $c(p)$ be defined similarly for each $Z(p^\infty)$. The author gives a necessary and sufficient condition for G to admit the structure of a compact topological group in terms of the above $c(R)$ and $c(p)$. It is also shown that the class of one-dimensional, connected and compact topological groups which are algebraically isomorphic with the reals mod 1 has the cardinal of the continuum. *K. Iwasawa* (Cambridge, Mass.).

Hartman, S.; et Hulanicki, A. Les sous-groupes purs et leurs duals. Fund. Math. **45** (1957), 71–77.

Let G be a locally compact abelian group and H a closed subgroup of G. H is called a pure subgroup of G if $nH=H\cap nG$ for every integer n, and H is said to have the property (P) if every character of H having a finite order can be extended to a character of G with the same order. If H is pure, the annihilator of H in the dual group of G has the property (P). The authors prove that when G or the dual of G is generated by a compact neighborhood of 1, or when G is the direct sum of a connected group and a discrete group without torsion, then a closed subgroup H of G is pure if and only if H has the property (P). *K. Iwasawa* (Cambridge, Mass.).

Żelazko, W. On the divisors of zero of the group algebra. Fund. Math. **45** (1957), 99–102.

If G is a locally compact topological group, the Banach space $L_1(G)$ of all functions on G integrable with respect to left invariant Haar measure μ is an algebra under convolution

$$f*g(\sigma)=\int_G f(\sigma\tau^{-1})g(\tau)d\mu(\tau).$$

The author shows that this algebra must contain divisors of zero. *K. de Leeuw* (Stanford, Calif.).

Lashof, Richard K. Lie algebras of locally compact groups. Pacific J. Math. **7** (1957), 1145–1162.

On a finite-dimensional Lie group, there is an analytic mapping from its Lie algebra onto a neighborhood of its identity which is even a homeomorphism. This is not necessarily true when the group is a generalized Lie group. Since every locally compact (connected) group is a generalized Lie group, or, as the author calls it, a *LP* group, its Lie algebra can be defined as a projective limit of Lie algebras of approximating groups. However, the exponential mapping is not open, in general. When the group under consideration is infinite-dimensional, this mapping even fails to be homeomorphic. K. Iwasawa proved [Ann. of Math. (2) **50** (1949), 507–558; MR **10**, 679] that every generalized Lie group has a neighborhood which is a direct product of a local Lie group and a compact normal subgroup. Using these results, the author shows with carefully checked details that a Lie algebra of a locally compact group is a direct product of a finite-dimensional Lie algebra, an abelian (possibly infinite-dimensional) Lie algebra, and a (possibly infinite) product of compact Lie algebras. These results hold for subgroups too. *H. Yamabe* (Minneapolis, Minn.).

Ionescu Tulcea, C. T. Sur certaines classes de fonctions de type positif. Ann. Sci. Ecole Norm. Sup. (3) **74** (1957), 231–248.

The main results obtained in this paper were announced by the author in C. R. Acad. Sci. Paris **243** (1956), 1389–1392 [MR **18**, 490]. *C. E. Rickart* (New Haven, Conn.).

Beck, Anatole. On invariant sets. Ann. of Math. (2) **67** (1958), 99–103.

The following theorem is proved: In a metric space, any closed set can be represented as the set of fixed points of a one-parameter group of transformations of the space if and only if the empty set can be so represented.

This was previously unknown even for the case of the plane. It indicates the probable futility of any attempt to classify the actions of non-compact transformations groups (even the simplest) on spaces of this type. This is particularly disconcerting in the case of such a nice space as the plane.

The proof, though ingenious, is elementary in the sense that it uses only simple properties of distance, continuity, and integration of real functions. *P. S. Mostert.*

Shields, Allen. The n-cube as a product semigroup. Michigan Math. J. **4** (1957), 165–166.

By the term semigroup we mean a Hausdorff space with a continuous associative multiplication. An (I)-semigroup is a semigroup on the unit interval in which 0 is a zero and 1 is an identity. This note consists of the following theorem. Let S be a semigroup on the n-cell ($n>1$), with boundary B. Let $J_1, \cdots, J_n$ be (I)-semigroups, and let $T=J_1\times\cdots\times J_n$ be the product semigroup on the n-cell with boundary C. If φ is a homeomorphism of C onto B such that $\varphi(c)\varphi(d)=\varphi(cd)$ whenever c, d and cd are in C, then S is isomorphic to T. *R. J. Koch.*

Koch, R. J. Note on weak cutpoints in clans. Duke Math. J. **24** (1957), 611–615.

A clan is a compact connected semigroup with unit. Since a clan need not be a homogeneous space, one may inquire as to the topological position of algebraically distinguished points, with some hope of a useful answer. Th. 1 is of a somewhat technical nature, identifying a topologically defined subset as a normal subgroup of H_u, the maximal subgroup of the clan which contains the unit, u. This leads to results germane to the conjecture that the unit of a clan which is not a group cannot be a weak cutpoint. The reviewer has conjectured that finite-dimensional homogeneous clans are necessarily groups. Th. 4 states that the unit of a homogeneous clan which is not a group (though perhaps infinite-dimensional) cannot be a weak cutpoint, and that there must be small opens sets about u with connected complements. Th. 5 is the result that, in a one-dimensional clan which is not a group, the unit cannot cut any subcontinuum.

A. D. Wallace (New Orleans, La.).

Keiner, Horst. Verallgemeinerte fastperiodische Funktionen auf Halbgruppen. Arch. Math. **8** (1957), 129–134.

The author extends Maak's definition of ω-functions and ω-almost-periodic functions [J. Reine Angew. Math. **190** (1952), 34–48; MR **13**, 910] from groups to semigroups, and generalizes the results of Maak that an ω-function is strongly ergodic and that an ω-almost-periodic function is the sum of an almost-periodic function and a zero-function. *E. Følner* (Copenhagen).

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: Day.

Lie Groups and Algebras

Wallace, E. W. A note on harmonic Lie groups. J. London Math. Soc. **33** (1958), 34–35.

The definition of a harmonic space was extended recently from Riemannian spaces to those with a symmetric affine connexion, and a Lie group is now said to be harmonic if it is harmonic with respect to its 0-connexion. It is proved that a Lie group is harmonic if and only if its Lie algebra is nilpotent, and examples are given to show that there are indecomposable harmonic Lie groups which are not Riemannian.

A. G. Walker (Liverpool).

See also: Topological Groups: Lashof.

Topological Vector Spaces

Mrówka, S. On the form of certain functionals. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1061–1067, LXXXVIII. (Russian summary)

The author studies the form of those real functionals ϕ defined on the set $C^*(X)$ of all bounded continuous real-valued functions on a completely regular Hausdorff space X that satisfy the following conditions: (1) $\phi(f+g)=\phi(f)+\phi(g)$; (2) if $f(p)\leqq g(p)$ for each $p\in X$, then $\phi(f)\leqq\phi(g)$; (3) if $f_n(p)\to f(p)$ for each $p\in X$, then $\phi(f_n)\to\phi(f)$. The main theorem is that each functional ϕ on $C^*(X)$ satisfying (1)–(3) has the form $\phi(f)=\sum_1^k \alpha_i f(p_i)$, where $p_1, \cdots, p_k\in X$ are fixed, and $\alpha_1, \cdots, \alpha_k$ are fixed real numbers, if and only if X is a Q-space [defined by Hewitt, Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 126]. This generalizes a result of Mazur [Ann. Soc. Polon. Math. **19** (1946), 241] on metric spaces. One of the preliminary results developed for use in the proof is a characterization of Q-spaces as those spaces X for which $p_0\in\beta X-X$ implies the existence of functions $f, f_1, f_2, \cdots$ in $C^*(\beta X)$ such that

$f_n(p) \to f(p)$ for each $p \in X$, but $f_n(p_0) \nrightarrow f(p_0)$. The possibility of replacing pointwise convergence by other types of convergence (in particular, by σ-uniform convergence) is discussed.

{Among several misprints, the two that might cause concern are on p. 1061, l. 3: Replace "$\aleph_1$" by "$\aleph_\lambda$". Related results have been obtained by M. Landsberg [Math. Ann. 132 (1956), 256–262; MR 18, 751].}

C. W. Kohls (Urbana, Ill.).

Klee, V. L., Jr. Extremal structure of convex sets. Arch. Math. 8 (1957), 234–240.

Let C be a linearly closed convex subset of a real linear space, which contains no line. The author establishes for such sets analogues of the classical theorems on extreme points of compact convex sets. An extremal ray of C is an open half-line $\rho \subset C$ such that $]x, y[\subset \rho$ whenever $]x, y[\subset C$ and $]x, y[\cap \rho$ is not empty. The union of all extremal rays (extreme points) of C is rext C (ext C); the closure (convex hull) of A is cl A (conv A). The main results are: 1. If C is finite-dimensional then $C = \text{conv}(\text{ext } C \cup \text{rext } C)$. 2. If C is a locally compact closed set in a locally convex Hausdorff space, then $C = \text{cl conv}(\text{ext } C \cup \text{rext } C)$. 3. If C is as in 2, and $X \subset C$, then cl conv $X = C$ if and only if cl $X \supset$ ext C, and for each extremal ray $\{x+ty: t>0\}$ of C, each $t>0$ and each neighbourhood V of $x+ty$, there is a point z in X such that $[x, z] \cap V$ is not empty.

The paper concludes with some results on the closedness or otherwise of ext C, rext C and related sets, for low-dimensional C. *J. H. Williamson* (Belfast).

Klee, V. L., Jr. Extremal structure of convex sets. II. Math. Z. 69 (1958), 90–104.

Continuing the notations of the preceding review, let C be a subset of a topological linear space. Let exp C be the set of exposed points of C, as defined by Straszewicz [Fund. Math. 24 (1935), 139–143], rexp C the set of exposed rays (defined in the present paper), sm C the smooth points of C.

It is proved that if C is a locally compact closed subset of a normed linear space, then ext $C \subset$ cl exp C and $C = \text{cl conv}(\exp C \cup \text{rexp } C)$. The relation between exp C and ext C is discussed, and partial results are obtained. The duality between the operators exp and sm is exploited to obtain results on exposed points; in particular, sufficient conditions for $C = \text{cl conv exp } C$. The final section describes a method of construction of some examples of convex bodies, via convex functions.

Throughout the paper various unsolved problems of general interest are mentioned; for example: (1) Characterize locally convex Hausdorff linear spaces in which every compact convex subset is the closed convex hull of its exposed points. (2) Must a bounded closed convex subset of a Banach space be supported at some point? *J. H. Williamson* (Belfast).

Pełczyński, A. On the approximation of S-spaces by finite dimensional spaces. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 879–881, LXXV. (Russian summary)

A B_0-space of S-type is a locally convex complete metrizable space X whose topology is defined by an increasing sequence of pseudonorms $\{|x|_n\}$ such that $\{x: |x|_{n+1} \leq 1\}$ is totally bounded in the pseudometric induced by $|\cdot|_n$. Let $\alpha_{ij}(X, t)$ be the maximal number of points in any subset S of $\{x: |x|_{i+j} \leq 1\}$ such that $|x-x'|_i \geq t$ whenever $x, x' \in S$, $x \neq x'$. Theorem: If B_0-spaces of S-type X, Y are isomorphic, then for each pair of sequences of pseudonorms that define the respective topologies, and for every $t>0$, i', j, there exist $\delta>0$, i, j', such that $\alpha_{ij'}(Y, t) \leq \alpha_{ij}(X, \delta t)$. The space of entire functions, the space of functions holomorphic in the interior of a circle, and the space of infinitely differentiable functions on a closed interval, each with a suitable sequence of pseudonorms, are of S-type. By the theorem, no two of them are isomorphic. *M. Jerison* (Princeton, N.J.).

Deprit, André. Une propriété des homomorphismes d'espaces de Fréchet. Acad. Roy. Belg. Bull. Cl. Sci. (5) 43 (1957), 834–837.

Let E and F be two Fréchet spaces (locally convex, complete and metrisable vector spaces over the reals). Let u_k, $k=1, 2$, be a homomorphism of E into F (a continuous linear mapping with closed range). It is shown that if $u_2(E)$ is finite-dimensional then u_1+u_2 is a homomorphism of E into F. *B. Yood* (New Haven, Conn.).

Weston, J. D. The principle of equicontinuity for topological vector spaces. Proc. Univ. Durham Philos. Soc. Ser. A. 13 (1957), 1–5.

Let X and X_0 be linear topological spaces over the reals, not necessarily locally convex. The principle of equicontinuity is said to hold for the ordered pair (X, X_0) if every bounded family F of continuous linear transformations of X into X_0 is equi-continuous. (F is called bounded if for each x in X the set $\{f(x) | f \in F\}$ is a bounded set in X_0.) The author proves the following version of the Banach-Steinhaus theorem for such a pair of spaces (X, X_0). Let $\{f_n\}$ be a sequence of continuous linear transformations of X into X_0 such that $\lim f_n(x) = f(x)$ exists for all x. Then f is continuous. If, in addition, X_0 is sequentially complete the sequence $\{f_n(x)\}$ is necessarily convergent everywhere if it is bounded for each x and convergent on a dense set in X. *B. Yood.*

Nef, Walter. Über monotone Linearformen, die im Lebesgueschen Sinne stetig sind. Arch. Math. 8 (1957), 334–335.

Let M be a partially ordered locally convex space over the real numbers. The supremum or infimum of a monotonic sequence $\{x_i\}$ $(i=1, 2, \cdots)$ is denoted by $\lim x_i$, if it exists. A linear functional f is called Lebesgue (or L-) continuous if $z = \lim x_i$ implies $f(z) = \lim f(x_i)$; the problem of existence and extension of such functionals is discussed by the following argument. A neighbourhood U of 0 is called selected if it is convex and symmetric, and if for each sequence x with $\lim x_i = 0$ all x after a certain point are in U. It is shown that the selected neighbourhoods are the basis of a topology (the L topology) on M. A monotonic linear form on M is continuous and Lebesgue continuous if and only if it is continuous in the L topology. Let P denote the cone of positive elements in M. Then for any x in M, there is a continuous and L-continuous form f with $f(x)=1$, if and only if $-x$ is not in the closure of P for the L topology; if f is a linear form on a subspace X of M, then f can be continued to a continuous and L-continuous form on M if and only if there is an L neighbourhood U of 0 such that f is bounded below on $X \cap (P+U)$. *J. L. B. Cooper* (Cardiff).

★ **Saltzer, Charles. The theory of distributions.** Advances in applied mechanics, Vol. V, pp. 91–110. Academic Press Inc., New York, N.Y., 1958. x+459 pp. $12.00.

This is an elementary expository account of the theory

of distributions as presented by Mikusiński and Temple. It gives applications to the Hadamard finite part of an integral, to Fourier series, Fourier transforms, weak solutions of ordinary differential equations, and finally to generalised harmonic analysis and statistical processes.

G. Temple (Oxford).

★ **Lighthill, M. J. Introduction to Fourier analysis and generalised functions.** Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1958. viii+79 pp. $3.50.

This short monograph, although intended for undergraduates, provides a really deep and searching account of Fourier series and integrals, including a systematic treatment of their asymptotic behaviour for large values of the argument. This has been made possible in the small compass of 79 pages by omitting the classical theory, with its delicate and difficult conditions, and basing the whole development on the Schwartz's theory of distributions, as simplified by Mikusiński and Temple. The species of test functions which are needed for this development are "good functions", i.e., functions of rapid decay at infinity, and "fairly good functions", i.e., functions of slow growth at infinity. The generalised functions correspond roughly to "distributions tempérées".

A chapter on generalised monomials, such as

$$|x|^{\alpha}=(\alpha+1)^{-1}\cdots(\alpha+n)^{-1}\frac{d^n|x|^{\alpha+n}(\operatorname{sgn} x)^n}{dx^n},$$

and their products with logarithms, furnishes a standard table of Fourier transforms and the raw material for their asymptotic estimation, which is determined by the singularities of the original function. In a final chapter, the author discusses Fourier series by similar methods. The author's claim that the use of generalised functions has superseded the old fashioned "summability" methods of handling divergent Fourier series is completely justified.

G. Temple (Oxford).

See also: Groups and Generalizations: Gravett. **Measure, Integration:** Matthes. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Andô.

Banach Spaces, Banach Algebras, Hilbert Spaces

Weston, J. D. The representation of linear functionals by sequences of functions. J. London Math. Soc. **33** (1958), 123–125.

The following consequence is obtained from a theorem of Helly. Let X be a normed vector space, and Y a separable subspace of the adjoint space X^*. To every bounded linear functional z on Y there corresponds a sequence $\{x_n\}$ of elements of X such that $\|x_n\|\leq\|z\|$ and, for every element y of Y, $z(y)=\lim y(x_n)$.

P. Civin.

Dinculeanu, Nicolae. Espaces d'Orlicz de champs de vecteurs. Fonctionnelles linéaires continues. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 269–275.

[Pour le parti I voir mêmes Rend. (8) **22** (1957), 135–139; MR **19**, 566.] Dans cet article l'auteur démontre deux théorèmes concernant certains espaces de Banach de champs de vecteurs définis sur un espace localement compact. Les espaces considérés généralisent les espaces L^p introduits par R. Godement [Ann. of Math. (2) **53** (1951), 68–124; MR **12**, 421], dans le même sens que les espaces d'Orliczd [voir, e.g., A. C. Zaanen Linear analysis, Interscience, New York, 1953; MR **15**, 878] généralisent les espaces L^p des functions numériquss de puissance p-ième intégrable par rapport à une mesure donnée. L'auteur démontre d'abord un théorème de décomposition qui généralise un résultat dû à John von Neumann [Ann. of Math. (2) **50** (1949), 401–485, Th. V; MR **10**, 548]. A l'aide de ce résultat, il obtient la forme explicite des fonctionnelles linéaires continues sur les espaces considérés.

C. Ionescu Tulcea (New Haven, Conn.).

Easterfield, T. E. Matrix norms and vector measures. Duke Math. J. **24** (1957), 663–669.

The author presents some results about norms on vector spaces (=vector measures) and the related norms on linear transformations of such spaces, presumably with a view to their applications to mathematical economics. Some statements tend to lack precision. Some results are already known. The result of 2.1 is incorrect in the infinite-dimensional case. The author's last result may be stated as follows: Let R be a ring of matrices of finite order, with a norm N, and V a vector space on which the matrices operate. Let $e\in V$ be such that $Re=V$. Then $M_N(x)=\inf_{Ae=x}N(A)$ is a norm on V.

J. H. Williamson (Belfast).

Kadison, Richard V.; and Singer, I. M. Three test problems in operator theory. Pacific J. Math. **7** (1957), 1101–1106.

The following problems, concerning the unitary equivalence of operators on Hilbert spaces, are discussed. Are B and C unitarily equivalent (u.e.), given that (1) $\begin{bmatrix}B&0\\0&B\end{bmatrix}$ and $\begin{bmatrix}C&0\\0&C\end{bmatrix}$ are u.e.; (2) $\begin{bmatrix}A&0\\0&B\end{bmatrix}$ and $\begin{bmatrix}A&0\\0&C\end{bmatrix}$ are u.e.; (3) B and CE are u.e., and C and BF are u.e., where $CE=EC$ and $BF=FB$? The results obtained are affirmative for (1) and (3), negative in general for (2). However, if the ring generated by $\begin{bmatrix}A&0\\0&B\end{bmatrix}$, and its commutant, are finite, the result is affirmative for (2) also. In the words of the authors, the proofs make use of some of the sophisticated techniques of rings of operators. In one or two places the proofs are indicated in outline only.

J. H. Williamson (Belfast).

Donoghue, W. F., Jr. The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation. Pacific J. Math. **7** (1957), 1031–1035.

The author gives four examples of various lattices of invariant subspaces. The transformations are all completely continuous and quasi-nilpotent. The first example is of a transformation T in Hilbert space whose invariant subspaces $\mathfrak{M}_n$ are linearly ordered; $\mathfrak{M}_n\supset\mathfrak{M}_{n+1}$, $n=0, 1, \cdots$. The adjoint transformation T^* then has an increasing set of finite-dimensional subspaces $\mathfrak{M}_n^*$ of dimension n. These examples show that the simplest possibility for invariant subspaces is actually realized. Another example adduces a transformation whose invariant subspaces form a continuous linearly ordered family, and a final one gives a transformation with a non-linearly ordered set of such subspaces.

E. R. Lorch (New York, N.Y.).

Gonshor, Harry. Spectral theory for a class of non-normal operators. II. Canad. J. Math. **10** (1958), 97–102.

In his previous paper [same J. **8** (1956), 449–461; MR **18**, 915] the author proved a spectral theorem for binormal

(or J_2) operators. In the present work he studies uniformly closed rings generated by a J_2 operator, obtaining a representation as complex- and second-order-matrix-valued functions on a spectrum space.

E. L. Griffin, Jr. (New York, N.Y.).

Krabbe, G. L. On the spectra of certain Laurent matrices. Proc. Amer. Math. Soc. **8** (1957), 894–897.

Let $1\leq p<\infty$, and let l_p denote the sequence space of all complex sequences $a=\{a_n\}_{-\infty}^{\infty}$ with $\sum_{-\infty}^{\infty}|a_n|^p<\infty$. Given a in l_1, the convolution $b*a$ belongs to l_p for each b in l_p, and the mapping $b\to b*a$ is a linear operator (denoted by a_{*p}) in l_p, the operator corresponding to the 'Laurent matrix' $(a_{n-\nu})$. It is here proved that the spectrum of this operator a_{*p} is the range $\{A(\theta):-\pi\leq\theta\leq\pi\}$ of the function $A(\theta)=\sum_{-\infty}^{\infty}a_n e^{in\theta}$. Related results have been proved by the author [same Proc. **7** (1956), 783–790; MR **18**, 587] in the case $p>1$.

F. F. Bonsall (Newcastle-upon-Tyne).

Andô, Tsuyoshi. Positive linear operators in semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1957), 214–228.

Let R be a partially ordered real Banach space having the property that every bounded subset has sup and inf. Suppose also that the norm of R has the property: if $|a|\leq|b|$ then $\|a\|\leq\|b\|$.

The author studies a positive completely continuous operator A acting on R, with the following property: if $\{a_\lambda\}$ is any directed set decreasing to zero, then $\inf_\lambda|Aa_\lambda|=0$ (A is positive means that A takes a positive element into a positive element of R). A linear manifold M of R is said to be normal if there exists a projection E of R into M such that $|x-Ex|\wedge|Ey|=0$ for every $x, y\in R$. The author defines the concept of complete positiveness for A, which is equivalent to the statement that 0 and R are the only normal manifolds invariant under A. The operator A is naturally decomposible if there is a decomposition of R into a direct sum of manifolds $\{N_\rho\}$ and M such that $A(N_\rho)\subset N_\rho$ for each ρ, $A(M)\subset M$, the restriction of A to each N_ρ is completely positive, and the restriction of A to M is a generalized nilpotent. The author gives conditions under which the operator A is naturally decomposible. He also gives the conditions that A is completely positive and that A is a generalized nilpotent.

The author concludes the paper with the following interesting theorem, which is a generalization of the results of G. Frobenius [S.-B. Preuss. Akad. Wiss. **1912**, 456–477] about matrices with non-negative elements: If the above space R is reflexive then the set of all eigenvalues λ of the positive completely continuous operator A such that $|\lambda|=\|A\|_s$ ($\|A\|_s$ denotes the spectral norm of A) coincides with the set of all kth roots of $\|A\|_s$, for $k=k_1, k_2, \cdots, k_n$, where $k_1, k_2, \cdots, k_n$ are some integers (it is understood that one has to consider the complex extension of R in order to get all eigen-vectors of A).

P. Saworotnow (Washington, D.C.).

Day, Mahlon M. Amenable semigroups. Illinois J. Math. **1** (1957), 509–544.

For any semigroup Σ, let $m(\Sigma)$ denote the Banach algebra of all continuous real-valued functions on Σ. An element μ of the conjugate space $m(\Sigma)^*$ of $m(\Sigma)$ is called a mean on $m(\Sigma)$ if $\inf\{x(\sigma):\sigma\in\Sigma\}\leq\mu(x)\leq\sup\{x(\sigma):\sigma\in\Sigma\}$ for all $x\in m(\Sigma)$. Invariance of a mean is defined in the usual way, and a semigroup is called amenable if there is an invariant mean on $m(\Sigma)$.

A large number of theorems are proved concerning existence of amenable semigroups. In particular it is shown that all abelian semigroups, solvable groups, and locally finite groups are amenable, and that no amenable group contains a free group on more than one generator. Moreover, subgroups, factor groups, group extensions, weak (but not full) direct products, and expanding unions of amenable groups are amenable.

It is shown that a semigroup is amenable if and only if it is strongly amenable. [This concept was introduced for groups by the author in Trans. Amer. Math. Soc. **69** (1950), 276–291; MR **13**, 357.] This result is used to improve a number of theorems in that paper concerning the relation between means and ergodicity.

Arens [Proc. Amer. Math. Soc. **2** (1951), 839–848; MR **13**, 659] showed how to define multiplication on the second conjugate space B^{**} of a Banach algebra B, and gave an example of a commutative Banach algebra B such that B^{**} is not commutative. The author shows that if $B=l_1(\Sigma)$ is the semigroup algebra of a semigroup Σ that has more than one invariant mean on $m(\Sigma)$, then $B^{**}=m(\Sigma)^*$ is not commutative. The semigroup of non-negative integers and any infinite abelian group have this property. Indeed, an abelian group admits a unique invariant mean if and only if it is finite.

Every finite group admits a unique invariant mean. If G is an infinite group that either is amenable and has an element of infinite order, or is locally finite, or whose chain of commutator subgroups ends at the identity in a finite number of steps, then G admits more than one invariant mean. Thus if the classical Burnside conjecture is true (i.e., if every finitely generated torsion group is finite), then every infinite torsion group admits more than one invariant mean.

G. G. Lorentz's characterization of almost convergent sequences [i.e., sequences of real numbers that are summed to the same value by all Banach limits; cf. Acta Math. **80** (1948), 167–190; MR **10**, 367] is generalized so as to hold for arbitrary amenable semigroups.

The paper closes with a discussion of what can be done if $m(\Sigma)$ is replaced by certain kinds of invariant closed linear subspaces X. (That is, one considers invariant means on X.) Particular attention is paid to the case when Σ is a topological space, and X consists of continuous functions. *M. Henriksen* (Lafayette, Ind.).

Nevanlinna, F. Über die Umkehrung differenzierbarer Abbildungen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **245** (1957), 14 pp.

This note extends a result of R. Nevanlinna [same Ann. no. **185** (1955); MR **16**, 806] on the inversion of a differentiable mapping between real Hilbert spaces R_x and R_y. It is assumed that: (A) $y:R_x\to R_y$ is differentiable on a sphere $|x|<\rho_0$ and $y(0)=0$; (B) $\inf\{|y'(0)h|;\ |h|=1\}=\mu>0$; (C) $\lim\sup_{x\to 0}|y'(x)-y'(0)|<\mu$; and (D) $y'(0)$ maps onto R_y. Two numbers ρ_x and ρ_y are found such that on the (open) image G_y of the sphere $|x|<\rho_x$, the function $y(\cdot)$ is uniquely invertible, the inverse function $x(\cdot)$ is differentiable, with $x'(y)=y'(x)^{-1}$, and such that G_y contains the sphere $|y|<\rho_y$. It is further shown that, in general, the numbers ρ_x, ρ_y cannot be increased. This extends previous results, in that it does not assume that y' is continuous even at $x=0$. *R. G. Bartle.*

Altman, M. On the approximate solutions of operator equations in L^p spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1099–1103, XCI. (Russian summary)

Szeptycki, P. A remark on the method of M. Altman of solving non-linear equations in L^p space. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1109–1112, XCI–XCII. (Russian summary)

Let P be a (non-linear) continuous map of a closed sphere S of $L^p(a, b)$ into L^p, where $p>1$. The functional equation $P(x)=0$ is to be solved by solving the real-valued equation $F(x)=\|P(x)\|^p=0$. An iterative technique is applied which is a modification of Newton's method previously developed by Altman [same Bull. **5** (1957), 457–460, 461–465; MR **19**, 984] and applied to the case $p=2$. Among other hypotheses, Altman assumes that the Fréchet derivative of P has a uniformly bounded inverse on S. The second paper under review calculates, for $p>2$, the second Fréchet derivative of $F(x)$, and expresses the conditions entirely in terms of P, its derivatives of first and second orders, and their bounds.

R. G. Bartle (Urbana, Ill.).

Kato, Tosio. On finite-dimensional perturbations of self-adjoint operators. J. Math. Soc. Japan **9** (1957), 239–249.

Let H_0, H_1, V be self-adjoint, V m-dimensional and $H_1=H_0+V$. It is proved that H_0, H_1 are unitarily equivalent except for singular separable parts with multiplicities not exceeding m. More precisely, there exist subspaces $\mathfrak{M}_0$, $\mathfrak{M}_1$, $\mathfrak{M}_{01}\subseteq\mathfrak{M}_0\cap\mathfrak{M}_1$ such that (i) $\mathfrak{M}_{01}$ reduces H_0, H_1 and their parts in $\mathfrak{M}_{01}$ are identical, (ii) the parts of H_0, H_1 in the separable $\mathfrak{H}\ominus\mathfrak{M}_{01}$ have spectra with multiplicities not exceeding m, (iii) for $i=0, 1$, $\mathfrak{M}_i$ reduce H_i, the parts of H_i in $\mathfrak{M}_i$ are unitarily equivalent, (iv) the parts of H_i in $\mathfrak{H}\ominus\mathfrak{M}_i$ are singular, (v) the parts of H_i in $\mathfrak{M}_i\ominus\mathfrak{M}_{01}$ are absolutely continuous. The subspaces and the corresponding projections P_i can be chosen so that the strong $\lim_{t\to\pm\infty}\exp(itH_1)\exp(-itH_0)=U_\pm$ exist as partially isometric operators with initial set $\mathfrak{M}_0$ and final set $\mathfrak{M}_1$. $U_\pm$ transform the self-adjoint H_0P_0 into H_1P_1. Similarly for the adjoint $U_\pm^*$.

František Wolf.

Kato, Tosio. Perturbation of continuous spectra by trace class operators. Proc. Japan Acad. **33** (1957), 260–264.

In this paper, the author shows the result of the previous review substantially holds when V is in the trace class. This is accomplished by means of the following inequality. If $H=\int\lambda E(d\lambda)$, $d\|E(\lambda)x\|^2/d\lambda\leq m^2$, then $\int_{-\infty}^{\infty}\|A\exp(-itH)x\|^2dt\leq 2\pi m^2\|A\|_2^2$, where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.

František Wolf.

See also: Measure, Integration: Bogdanowicz; Gordon and Lorch. **Sequences, Series, Summability:** Devinatz. **Ordinary Differential Equations:** Feller. **Partial Differential Equations:** Browder. **Topological Groups:** Żelazko; Ionescu Tulcea. **Numerical Methods:** Schäfke. **Probability:** Šidák. **Quantum Mechanics:** Albertoni and Duimio.

TOPOLOGY

General Topology

Dekker, T. J.; and de Groot, J. Decompositions of a sphere. Fund. Math. **43** (1956), 185–194.

The authors sharpen and extend a result of R. M. Robinson [Fund. Math. **34** (1947), 246–260; MR **10**, 106]. They prove that the sphere S can be decomposed into $m\leq c$ mutually disjoint non-empty subsets, satisfying any given number $n\leq c$ of congruences between non-empty and non-complete (i.e., whose sum does not equal S) — but otherwise arbitrary finite or infinite — sums of the subsets. Further, if m, $n<c$, all the subsets can be chosen to be totally imperfect. Various consequences of this paradoxical result are discussed and some unsolved problems are stated.

P. Erdős (Toronto, Ont.).

Dekker, Th. J. Decompositions of sets and spaces. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 581–589, 590–595.

In this paper the author extends the result proved in the previous review, for the two-dimensional sphere, to the n-dimensional sphere ($n\geq 3$), except for $n=4$, for which case the problem is unsolved. It is not known if the same result holds for Euclidean space of dimension n ($n\geq 3$); it does not hold for $n=2$. The validity of the theorem for hyperbolic and elliptic spaces is also discussed. Various further problems and results are discussed on decompositions of spaces, on permutation groups, and groups of matrices. The author obtains his results from a general decomposition theorem which is of independent interest, but is fairly lengthy to state.

P. Erdős.

Dekker, Th. J. Decompositions of sets and spaces. III. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 104–107.

The author continues his studies of the decompositions of Euclidean, hyperbolic and elliptic spaces. Many interesting unsolved problems are stated.

P. Erdős.

Viola, Tullio. Construction et propriétés de certains ensembles de points sur le plan hyperbolique ou sur la droite projective. C. R. Acad. Sci. Paris **243** (1956), 1819–1821.

The author poses the following problem: A transfinite number $\alpha\geq\omega$ of the second class being arbitrarily given, to construct in the half-plane $I(z)>0$ (alternatively in the X-axis) a denumerable point set E_α, containing a transfinite sequence $z_0, z_1, \cdots, z_{\alpha'}, \cdots$ (with $1\leq\alpha'<\alpha$), such that, whatever the suffix $\alpha'<\alpha$ may be, one has $E_\alpha\simeq E_\alpha-\mathbf{U}_{0\leq\xi<\alpha'}(z_\xi)$. A solution of this problem is given for the case $\alpha=\omega^2$.

B. Germansky (Jerusalem).

Mrówka, S. Some remarks on compactness. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 799–801, LXVIII. (Russian summary)

The following "Borel covering theorem", in terms of general set theory, was formulated by Veress [Acta Litt. Sci. Szeged **6** (1932), 34–45]:

Let $\{U_\alpha\}$ be a countable family of subsets (of a set X) such that for each $A\subset\mathbf{U}_\alpha U_\alpha$ with $\bar{\bar{A}}=\aleph_0$ there exists $U\in\{U_\alpha\}$ with $\overline{\overline{A\cap U}}=\bar{\bar{A}}$. Then $\{U_\alpha\}$ contains a finite subfamily $\{U_{\alpha_i}\}$ with $\mathbf{U}_{i=1}^n U_{\alpha_i}=\mathbf{U}_\alpha U_\alpha$.

This paper gives results which are generalizations of, or related to, the above theorem. For example, the author

shows that if $\{U_\alpha\}$ is a family of cardinality m (m infinite) such that for each $A \subset \mathbf{U}_\alpha U_\alpha$ with $\bar{\bar{A}} = m$ there exists $U \in \{U_\alpha\}$ with $\overline{\overline{A \cap U}} = \bar{\bar{A}}$, then $\{U_\alpha\}$ contains a subfamily $\{U_{\alpha_\beta}\}$ such that $\overline{\overline{\{U_{\alpha_\beta}\}}} < m$ and $\mathbf{U}_\beta U_{\alpha_\beta} = \mathbf{U}_\alpha U_\alpha$. A condition is also given for $\{U_\alpha\}$ to contain a finite subfamily $\{U_{\alpha_n}\}$ such that $\mathbf{U}_n U_{\alpha_n} = \mathbf{U}_\alpha U_\alpha$.

Related results by Sierpiński and Ju. Smirnov are also mentioned. *H. Tong* (Middletown, Conn.).

Motchane, Léon. Sur la notion d'espace bitopologique et sur les espaces de Baire. C. R. Acad. Sci. Paris **244** (1957), 3121–3124.

Motchane, Léon. Sur la caractérisation des espaces de Baire. C. R. Acad. Sci. Paris **246** (1958), 215–218.

The second paper is a correction of the first. The definitions are too complicated to reproduce here. The author shows that for a topological space to be everywhere second category, it suffices to require that in every open subset Ω, certain sequences of sets dense in Ω but not necessarily open (open in a stronger topology) satisfy a condition weaker than having a non-empty intersection (namely convergence in a third topology). The condition is shown to be necessary as well as sufficient. *J. Isbell.*

Kerstan, Johannes. Zur Charakterisierung der pseudokompakten Räume. Math. Nachr. **16** (1957), 289–293.

Seven statements equivalent to pseudo-compactness for a completely regular topological space are obtained, along with 13 statements equivalent to pseudo-compactness for an arbitrary topological space. Several of these have also been given independently by Iseki and Kasahara [Proc. Japan Acad. **33** (1957), 100–102; MR **19**, 668]. Sample assertions: A completely regular space is pseudo-compact if and only if every star-finite open covering is finite. A topological space X is pseudo-compact if and only if every locally finite covering of X by sets of the form $f^{-1}(H)$ (f is continuous and real-valued on X, H is an open set of real numbers) is finite. *E. Hewitt* (Seattle, Wash.).

Isiwata, Takeshi. On a completely regular space X and $T(X)$. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 227–236.

The author studies the set $T(X)$ of all completely regular topologies on the underlying point set of X that are weaker than the topology of X. Typical results are the following. (1) If X is not compact, then the cardinal of $T(X)$ is not less than that of X. (2) If X is locally compact but not compact, then the number of minimal (=compact) members of $T(X)$ is not less than the cardinal of X. (3) $T(X)$ consists exclusively of locally compact topologies if and only if $\beta X - X$ is finite. *M. Jerison.*

Hanner, O. Locally normal spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1051–1054, LXXXVII. (Russian summary)

A construction is given which imbeds any (regular) locally normal T_1 space in a normal T_1 space with one more point. This gives a new proof of the characterization [Čech, Ann. of Math. (2) **38** (1937), 823–844] of open subsets of normal T_1 spaces as locally normal spaces. The author remarks that every topological space is, in a trivial way, an open subset of a non-T_1 normal space, thus answering a question of Čech [ibid.]. *A. H. Stone.*

de Vries, H. Compactification of a set which is mapped onto itself. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 943–945, LXXIX. (Russian summary)

The author proves that the continuum hypothesis is equivalent to the assertion: Given a set S of cardinality $\mathfrak{c}$, and a 1-1 mapping ϕ of S onto itself, there exists a compact metric topology on S for which ϕ is a homeomorphism. In general, S cannot be made zero-dimensional as well. *A. H. Stone* (Manchester).

Mrówka, S. A remark on compactifications of a set. Bull. Acad. Polon. Sci. Cl. III. **5**(1957), 1105–1108, XCI. (Russian summary)

[Cf. the paper reviewed above.] Let S be a set with cardinality $|S| \geqq \mathfrak{c}$, and let ϕ be a 1-1 mapping of S onto itself. The author shows, without the continuum hypothesis, that there is a compact Hausdorff (but, in general, non-metric) topology on S for which ϕ is a homeomorphism. If instead $|S| = \aleph_0$, this becomes false; whether it is true when $|S| \geqq \aleph_1$, remains undecided. *A. H. Stone.*

Aronszajn, N.; and Panitchpakdi, P. Correction to: "Extension of uniformly continuous transformations in hyperconvex metric spaces". Pacific J. Math. **7** (1957), 1729.

Withdrawal of an incidental remark in same J. **6** (1956), 405–439 [MR **18**, 917].

See also: Measure, Integration: Abian. **Topological Groups:** Hulanicki; Beck. **Probability:** Nasr.

Algebraic Topology

Steenrod, N. E. Cohomology operations derived from the symmetric group. Comment. Math. Helv. **31** (1957), 195–218.

Steenrod discovered the squaring operations which now bear his name in 1946 [see Ann. of Math. (2) **48** (1947), 290–320; MR **9**, 154]. About 1949 he began to develop the theory of the reduced pth power operations for all primes p [see Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. I, Amer. Math. Soc., Providence, R.I., 1952, p. 530]. In case $p=2$ these operations reduce to the previously defined squaring operations. Steenrod's first attempt at a systematic account of these new operations was not published until 1952 [Ann. of Math. **56** (1952), 47–67; MR **13**, 966], although the theory had already been used by A. Borel and J.-P. Serre [C. R. Acad. Sci. Paris **233** (1951), 680–682; MR **13**, 319] to obtain some interesting results. In 1953 Steenrod published two brief notes [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 213–217, 217–223; MR **14**, 1005, 1006] which gave an outline of a new method of defining the reduced pth powers and proving their basic properties. Since the publication of these notes, he has further perfected this new method. The present paper and a successor by Steenrod and E. Thomas [see the following review] contain a full account of his latest results in this direction.

Roughly speaking, if G is any permutation group, then each homology class of G determines a cohomology operation according to the method given in 1953 [loc. cit.] by Steenrod. Of course there are many relations of dependence among the cohomology operations thus defined. For example, if G is of degree n, then a homology class of G gives the same operation as does its image in the full symmetric group of degree n. The construction given

in the present paper for cohomology operations based on the symmetric group of degree n is an improvement over that given previously: it gives more cohomology operations than the previous construction (e.g., the Pontrjagin squaring operation). The author also proves that the totality of cohomology operations based on the full symmetric group of degree n is generated by the subset consisting of those operations based on cyclic permutation groups of prime orders and degrees (this result is further refined in the paper reviewed below). *W. S. Massey.*

Steenrod, N. E.; and Thomas, Emery. Cohomology operations derived from cyclic groups. Comment. Math. Helv. **32** (1957), 129–152.

Steenrod, in the paper reviewed above, has defined a family of cohomology operations, called reduced powers, which are associated with elements of the cohomology group (in a generalized sense) of a certain permutation group. He showed that these operations are generated, with respect to composition, by four primitive operations, namely addition, cup-product, homomorphisms induced by coefficient homomorphisms, and the Bockstein operations, together with those reduced powers associated with cyclic permutation groups having degree p and order p where p ranges over the primes. In this paper this result is very much sharpened by obtaining a smaller generating set consisting of the (Steenrod) cyclic reduced powers $\mathcal{P}^i$ and the Pontryagin pth powers $\mathfrak{P}_p$ [see Thomas, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 266–269; MR **18**, 57].

Let $\theta=0$ or a prime power, let $m=0$ or a prime power, let p be a prime and let q, r be integers. Let M be the cochain complex having two free generators u, v in dimensions q, $q+1$ with $\delta u=\theta v$, let M^p be the p-fold tensor product of M, let π be the cyclic group of order p operating on M^p by cyclic permutations of the factors and let W be a π-free acyclic π-complex. Then any element ξ of $H^r(W\otimes_\pi M^p\otimes Z_m)$ determines a cyclic reduced power operation $\xi\colon H^q(K; Z_\theta)\to H^r(K; Z_m)$ and Steenrod's results show that such operations ξ, together with the primitive operations, generate all reduced powers. Thus the problem is to compute the cohomology of $W\otimes_\pi M^p\otimes Z_m$ and to interpret the elements of the preferred systems of generators. The computations proceed by distinguishing particular cases and are too detailed to be summarized here. *P. J. Hilton* (Ithaca, N.Y.).

Dugundji, J. Products in homotopy and homology groups. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 987–988.

If Y is a Hausdorff space and $y_0\in Y$, the author defines the reflectant $R(Y, y_0)$ to be the space obtained from $Y\times Y$ by identifying (y, y_0) with (y_0, y). He remarks that Y is an H-space with unit y_0 if and only if it is a retract of $R(Y, y_0)$. He defines products $\pi_p(Y)\otimes\pi_q(Y)\to\pi_{p+q}(R, Y)$ (the "principal pairing") and $H_p(Y)\otimes H_q(Y)\to H_{p+q}(R)$, and remarks that the Whitehead product is obtained by composing the principal pairing with the boundary homorphism $\partial\colon\pi_{p+q}(R, Y)\to\pi_{p+q-1}(Y)$. Using this fact, the author gives simple proofs of some well-known facts about Whitehead products. He defines several other pairings, one of which, the "composition pairing", seems to be a generalization, to arbitrary H-spaces, of a product studied by Moore in the case of loop spaces [Séminaire Henri Cartan Ecole Norm. Sup., 1954–1955, 2ème éd., 1956, Exp. 22; MR **19**, 439]. He gives some relations among these pairings and promises applications in a later paper. *N. Stein* (New Haven, Conn.).

★ **Milnor, John. Isotopy of links. Algebraic geometry and topology.** A symposium in honor of S. Lefschetz, pp. 280–306. Princeton University Press, Princeton, N. J., 1957. $7.50.

Let M be an open, orientable, triangulable three-dimensional manifold (e.g., Euclidean 3-space), and let C_n denote a space consisting of n disjoint circles. By an n-link $\mathfrak{L}$ in M is meant a homeomorphism $\mathfrak{L}\colon C_n\to M$. The image $\mathfrak{L}(C_n)$ is denoted by L. Two links $\mathfrak{L}$ and $\mathfrak{L}'$ are isotopic if there exists a continuous family of homeomorphisms $h_t\colon C_n\to M$ for $0\leq t\leq 1$ such that $h_0=\mathfrak{L}$ and $h_1=\mathfrak{L}'$. For polygonal links in Euclidean 3-space, K. T. Chen has proved [Ann. of Math. (2) **56** (1952), 343–353; MR **14**, 193] that the fundamental group of $M-L$, $F(M-L)$, modulo its qth lower central subgroup $F_q(M-L)$, is invariant under isotopy of $\mathfrak{L}$ for any $q>0$.

In this paper Chen's result is extended in three directions. (1) The restriction to polygonal links is removed. (2) A corresponding result is proved for links in an arbitrary open, orientable, triangulable 3-manifold M. In fact, let $K(M-L, M)$ denote the kernel of the inclusion homomorphism $F(M-L)\to F(M)$, and let $K_q(M-L, M)$ be its qth lower central subgroup. Then it is proved that the quotient group $F(M-L)/K_q(M-L, M)$ is invariant under isotopy of $\mathfrak{L}$. (Note that if M is Euclidean space, then $K(M-L, \mathrm{M})=F(M-L)$.) (3) Certain special elements of the group $F(M-L)/K_q(M-L, M)$ are singled out, called the "meridians" and "parallels" to the components of $\mathfrak{L}$. It is proved that their conjugate classes are invariant under isotopy of $\mathfrak{L}$.

The author then uses these results to construct certain numerical isotopy invariants for links in Euclidean space. It is first shown that the ith parallel β_i in $F(M-L)/F_q(M-L)$ can be expressed as a word w_i in the meridians $\alpha_1, \cdots, \alpha_n$. Let $\mu(i_1, \cdots, i_r)$ denote the coefficient of $\kappa_{i_1}, \cdots, \kappa_{i_{r-1}}$ in the Magnus expansion of w_{i_r} (where $\kappa_i=\alpha_i-1$). Then it is shown that the residue class $\bar{\mu}$ of $\mu(i_1, \cdots, i_r)$, modulo an integer $\Delta(i_1, \cdots, i_r)$ determined by the μ, is an isotopy invariant of $\mathfrak{L}$. It is also shown that $\bar{\mu}(i_1, \cdots, i_r)$ is a homotopy invariant of $\mathfrak{L}$ in a sense previously defined by the author [Ann. of Math. (2) **59** (1954), 177–195; MR **17**, 70] whenever the indices $i_1, \cdots, i_r$ are all distinct.

At the end of the paper, the author gives some examples of links for which he has computed the invariants $\bar{\mu}$. An example is given for which all the invariants $\bar{\mu}$ are zero, but which is not isotopically trivial, at least under isotopies which satisfy a certain smoothness condition. *W. S. Massey* (Providence, R.I.).

Weier, Joseph. Sopra un problema di dimensione. Rend. Mat. e Appl. (5) **15** (1956), 93–101.

Let f be a mapping $Q\to S$, where S is an n-sphere and Q a compact simplicial manifold of dimension $n+1$. Let p be a point of Q. Theorem: There exists a mapping g homotopic to f such that $g^{-1}(p)$ is either empty or is a simplicial 1-sphere. *P. A. Smith* (New York, N.Y.).

Weier, Joseph. La dimensione di certe soluzioni topologiche. Rend. Mat. e Appl. (5) **15** (1956), 329–340 (1957).

L'A. prosegue una sua ricerca [vedi la recensione precedente]: se Q è una varietà euclidea finita orientabile e di dimensione superiore a 3, immersa in uno spazio euclideo R; se S è una sfera di R, con dim Q=(dim S)+1; se a è un punto fisso di S ed F una classe di trasformazioni essenziali di Q in S omotopiche fra di loro: in F esiste una tal

trasformazione h, che l'insieme $h^{-1}(a)$ sia una sfera unidimensionale, D, la quale abbia grado 1 nella h; risulta poi dim $f^{-1}(a)\geq 1$ per ogni f di F, se D è, rispetto a Q, algebricamente rilevante. *G. Scorza-Dragoni* (Padova).

Weier, Joseph. Dimensione ed essenzialità. Ann. Univ. Ferrara. Sez. VII. (N.S.) **6** (1956–1957), 1–12.

Let f be a transformation from a set P into a set Q. Denote a k-dimensional sphere by S^k. (i) If P and Q are compact topological varieties such that the dimension m of P exceeds the dimension n of Q which exceeds one, if the homotopy group of the continuous transformations of S^{m-1} into S^{n-1} is trivial, and if f is an essential transformation, then the set $f^{-1}q$ is infinite for every point q in Q. (ii) If P is S^n, Q is S^{n-1}, n exceeds six, then there exists an essential transformation f for which there is at least one point q in Q such that $f^{-1}q$ is finite. (iii) If P is an open set in n-dimensional euclidean space, Q is S^{n-1}, n exceeds six, and if f is an essential transformation, then the dimension of the set $f^{-1}q$ is at least one for every point q in Q. (iv) If P is a closed two-dimensional variety, Q a circumference, and f is an essential transformation, then the dimension of $f^{-1}q$ is at least one for every q in Q. *P. V. Reichelderfer.*

Dyer, Eldon. Regular mappings and dimension. Ann. of Math. (2) **67** (1958), 119–149.

Based on G. T. Whyburn's n-regular convergence [Fund. Math. **25** (1935), 408–426] the author defines n-regular functions and investigates them and the cohomology sequences they induce. More precisely a mapping $f:X\to Y$ is called proper if for each compact set C of Y, $f^{-1}(C)$ is compact. A proper mapping f is said to be n-regular at the point y in Y if f takes any neighborhood of a point of $f^{-1}(y)$ onto a neighborhood of y, and for each x in $f^{-1}(y)$ and closed neighborhood U of x, there exist closed neighborhoods V of x and W of y such that $V\subset U$ and for each point z in $(W-y)$ the natural homomorphism of $\check{H}^i(U\cap f^{-1}(z))$ into $\check{H}^i(V\cap f^{-1}(z))$ is trivial for $i\leq n$.

He is able to find generalizations of known theorems on cohomologically locally connected spaces, and mappings which are n-regular at a point. There are also applications to dimension theory including the following modification of Boltyanskiĭ's characterization of dimensionally full valued compacta [Dokl. Akad. Nauk. SSSR (N.S.) **67** (1949), 773–776; MR **11**, 195]. Theorem: If an n-dimensional compactum X is lc^n (over the integers), then it is dimensionally full valued if and only if its cohomological dimension (over the rationals) is n. (Here, X dimensionally full valued means $\dim X\times Y=\dim X+\dim Y$ for each compactum Y.)

Haskell Cohen (Baton Rouge, La.).

GEOMETRY

Geometries, Euclidean and Other

Hansen, H. H. The congruence theorems of elementary geometry. Nordisk Mat. Tidskr. **5** (1957), 181–183, 213. (Danish. English summary)

Thybaut, A. Sur les hélices d'un complexe linéaire. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 884–897.

Lauffer, R. Elementare Sätze über Teilverhältnisse. Math.-Phys. Semesterber. **5** (1957), 289–292.

Kurepa, Đuro. On the symmetry. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 239–254. (Serbo-Croatian summary)

Some elementary results of which the following is typical: If A is an "algebraic set" in euclidean space R_n and if the set S_0A of all its centres of symmetry is non-empty, then S_0A is a subspace R_k of R_n $(0\leq k\leq s)$; if $k\geq 1$, A consists of a family of pairs of subspaces R_k parallel to and symmetrical with respect to S_0A (which may include the pair coinciding with S_0A). {Reviewer's note: The author's "algebraic set" is defined as a set A of points satisfying an equation $\sum a_{i_1\ldots i_n}x^{i_1}\cdots x^{i_n}=0$, but the following property of A is sufficient: if A has an infinity of points in common with a line L, then $L\subset A$.}

F. A. Behrend (Zbl **66** (1957), 404).

Rosati, Luigi Antonio. L'equazione delle 27 rette della superficie cubica generale in un corpo finito. I, II. Boll. Un. Mat. Ital. (3) **12** (1957), 612–626; **13** (1958), 84–89.

If F denotes a non-singular cubic surface of a linear 3-dimensional space over a field γ, of characteristic $p\neq 2$, then all possible cases concerning the number and the configuration of the lines over γ lying on F have been determined [B. Segre, same Boll. (3) **4** (1949), 223–228; MR **11**, 537]. Now the question is deeply investigated in the case when γ is finite, of order $q=p^h$ say. It is shown that the equation of the 27 lines of F, and the equation of the 45 tritangent planes of F, are then both always reducible over γ. More precisely, the former equation breaks up into factors, irreducible over γ, in one of the following 20 different manners (where $(m)_n$ indicates the existence of n irreducible factors of degree m, and $(m)_1$ is written simply as m): $\{(1)_{27}\}$, $\{(1)_{15}, (2)_6\}$, $\{(1)_9, (3)_6\}$, $\{(1)_7, (2)_{10}\}$, $\{(1)_5, 2, (4)_5\}$, $\{(1)_3, (2)_{12}\}$, $\{(1)_3, (2)_3, (3)_4, 6\}$, $\{(1)_2, (5)_5\}$, $\{1, (2)_3, (4)_5\}$, $\{1, (2)_4, (3)_2, (6)_2\}$, $\{2, (5)_3, 10\}$, $\{(3)_9\}$, $\{(3)_5, (6)_2\}$, $\{3, (6)_4\}$, $\{1, 2, (8)_3\}$, $\{(1)_3, (4)_6\}$, $\{(1)_3, (2)_3, (6)_3\}$, $\{3, (12)_2\}$, $\{1, (4)_2, 6, 12\}$, $\{(9)_3\}$.

The question of actual existence remains undecided for the last three cases; but it is shown that each of the 17 first possibilities may in fact occur. For the first 15 cases this is proved, for arbitrary $p\neq 2$ and sufficiently large h, by conveniently constructing certain cubic surfaces F birationally represented on γ upon a plane. As for the other cases, it is shown that the cubic surface $z(x^2-2y^2)+(y^2+z^2)t+2(x+y+z)t^2+t^3=0$ presents the 15th or the 16th case if $\gamma=\mathrm{GF}(5^k)$ or $\gamma=\mathrm{GF}(5^{2k})$, respectively, where k denotes any sufficiently large odd integer; and that the cubic surface $2xyz+(x^2+y^2-z^2)t+(3x+y)t^2-t^3=0$ presents the 17th case if $\gamma=\mathrm{GF}(7^k)$, where k denotes any sufficiently large odd integer not divisible by 3.

The proof of the stated results gives incidentally that the degree of the minimum extension $\mathrm{GF}(Q)$ of $\gamma=\mathrm{GF}(q)$ where F contains 27 lines, and so [Rosati, ibid. (3) **11** (1956), 412–418; MR **18**, 512] is such that the number of points of F in $\mathrm{GF}(Q)$ equals Q^2+7Q+1, is never greater than 12. *B. Segre* (Rome).

Matschinski, Matthias. Remarques sur les polytopes saturés. C. R. Acad. Sci. Paris **246** (1958), 528–531.

The author continues his work on regular polytopes, regarding them as configurations [cf. Coxeter, Regular

polytopes, Menthuen, London, 1948, pp. 12, 132, 144; MR **10**, 261]. In his description of the numbers n, μ, λ, q he comes close to rediscovering the Schläfli symbol $\{n, \mu\, \lambda, q, \cdots\}$. *H. S. M. Coxeter* (Toronto, Ont.).

Matschinski, Matthias. De la géométrie combinatoire et de la construction combinatoire des polytopes. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 543–568. (2 plates).

After describing the hypercubes γ_n and the cross polytopes β_n (which are attributed to the American, Stringhem, instead of the Swiss, Schläfli), the author considers maps on various surfaces, including one of 18 triangles on a torus (Figures 3a, 3b) which is the regular map $\{3, 6\}_{3,0}$ [Coxeter, Rep. Math. Colloq. (2) **8** (1949), 18–38; MR **10**, 616]. *H. S. M. Coxeter* (Toronto, Ont.).

Pâquet, P. V. Sur l'utilisation des opérateurs de projections dans l'enseignement de la géométrie vectorielle. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 898–919.

Busulini, Bruno. Sul gruppo delle pseudo-congruenze. Ann. Univ. Ferrara. Sez. VII. (N.S.) **6** (1956–1957), 27–39.

A similitude π in a one-dimensional complex form L is defined as the mapping $y=\lambda x+q$ $(\lambda\neq 0)$; π and π' are equivalent, $\pi\sim\pi'$, if $\lambda=\lambda'$. The mapping is a congruence if $\lambda=\pm 1$, direct if $\lambda=1$, inverse if $\lambda=-1$. Pairs of similitudes π_L and π_M $(y=\mu x+p)$ in two one-dimensional complex forms L, M represent similitudes $S=(\pi_L, \pi_M)$ in the plane of two complex dimensions. If π_{LM} represents a similitude from L to M, π_{ML} one from M to L, then $T=(\pi_{LM}, \pi_{ML})$ represents an inverse similitude in the plane. The author defines $S\sim S'$ if $\pi_L\sim\pi_L'$, $\pi_M\sim\pi_M'$; $T\sim T'$ if $T^2\sim T'^2$; $S\approx S'$ if $\pi_L\sim\pi_M'$, $\pi_M\sim\pi_L'$; $T\approx T'$ if $T\sim T'$. He states relations between these "equivalences". All S that satisfy the condition $S\approx S^{-1}$ and $\lambda\mu=1$ represent the group of direct congruences. All mappings of the form $X=\pm\lambda^{-1}x+q$, $Y=\lambda y+p$, and $X=\lambda y+p$, and $Y=\pm\lambda^{-1}x+q$ form the group of pseudo-congruences. Primitive invariance relations for these groups are stated. For a similitude to be real one has the conditions $\mu=\bar{\lambda}$, $q=\bar{p}$. Hence every real pseudo-congruence is a congruence.

H. Schwerdtfeger (Montreal, Que.).

Al-Dhahir, M. W. On a theorem of H. F. Baker. Proc. Math. Phys. Soc. Egypt **5** (1956), no. 4, 57–60 (1957). (Arabic summary)

The following theorem is a generalization of a theorem of Möbius [J. Reine Angew. Math. **3** (1828), 273–278] concerning Pappus' theorem and mutually inscribed tetrahedra, as well as the converse of this generalization: In an n-dimensional projective space $(n>1$, odd), defined over a field F, let $0, 1, 2, \cdots, n$ be points, no n of which are dependent. Let $0', 1', 2', \cdots, n'$ be any other set of points such that the following $n(n+1)/2$ congruences hold: $(0'1'23\cdots n)$, $(0'12'3\cdots n)$, $(0'123'\cdots n)$, $\cdots$, $(0123\cdots(n-2)(n-1)'n')$. Then the commutativity of F is a necessary and sufficient condition for the dependence of $0', 1', 2', \cdots, n'$.

The sufficiency of the condition has been established by H. F. Baker [Principles of geometry, vol. IV, Cambridge, 1925], but the present paper replaces Baker's analytic proof by synthetic considerations. A proof for the necessity has been known only for $n=3$ [Reidemeister, Jber. Deutsch Math. Verein. **40** (1931), Abt. 2, 48; Schönhardt, ibid. **40** (1931), Abt. 2, 48–50]. The paper deals also with a modified form of the theorem which yields, in a special case, a similar criterion for the commutativity of fields in connection with even-dimensional projective spaces.

R. Artzy (Haifa).

Lingenberg, Rolf. Zur Einführung von Koordinaten in einer projektiven Ebene mit Hilfe von Endomorphismen transitiver Translationsgruppen. Math. Z. **67** (1957), 332–360.

Die Artinsche Koordinateneinführung in einer affinen Ebene, bei der die Koordinaten als Endomorphismen der Translationsgruppe aufgefaßt werden, ist ihrem Wesen nach an die Existenz einer transitiven Translationsgruppe gebunden. Das Ziel der vorliegenden Arbeit ist es, eine entsprechende Koordinateneinführung auch in allgemeineren affinen Ebenen zu geben. Zu diesem Zweck werden anstelle der Translationen und Streckungen nach Auszeichnung von zwei Achsen und Einheitspunkt gewisse Projektivitäten der x-Achse betrachtet, die sich nach Vertauschen der Achsen als die Rechtsadditionen und Linksmultiplikationen der dualen Ebene im Sinne des Pickertschen Buches [Projektive Ebenen, Springer, Berlin, 1955; MR **17**, 399] erweisen. Die Menge $\mathfrak{T}$ aller Rechtsadditionen und die Menge $\mathfrak{H}$ aller Linksmultiplikationen bilden je eine Loop. Als Koordinaten dienen die Endomorphismen α von $\mathfrak{T}$, welche die Rechtsaddition $x\to x+t$ abbilden auf die Rechtsaddition $x\to x+\alpha t$. Ist τ dasjenige Element aus $\mathfrak{T}$, das den Achsenschnittpunkt in den Einheitspunkt der x-Achse überführt, so wird durch $\tau^{\alpha+\beta}=\tau^{\alpha}\tau^{\beta}$ und $\tau^{\alpha\beta}=(\tau^{\alpha})^{\beta}$ in der Endomorphismenmenge $\mathfrak{A}$ eine Addition und eine Multiplikation erklärt, mit der $\mathfrak{A}$ bzw. $\mathfrak{A}-0$ isomorph zu $\mathfrak{T}$ bzw. $\mathfrak{H}$ ist. Vorausgesetzt wird dabei die der Pickertschen Zerlegbarkeitsbedingung entsprechende Eigenschaft der Ebene. Im Rahmen dieser Beschreibung werden die bekannten Kennzeichnungen der Ebenen über Cartesischen Zahlsystemen, Rechts- und Linksquasikörpern und Fastkörpern neu hergeleitet [vgl. Pickert, loc. cit., § 3.5]. Soweit sie existieren, werden dabei konsequent die Translationen und Streckungen verwendet, welche die Elemente von $\mathfrak{T}$ und $\mathfrak{H}$ induzieren. Rechnen mit Koordinaten wird vermieden. Eine sorgfältige Untersuchung der gegenseitigen Beziehungen zwischen verschiedenen zentralen Kollineationen und Dualitäten führt u.a. auf die folgenden Sätze: (1) Ist die Gruppe $\mathfrak{P}_{u,v}$ der zentralen Kollineationen mit Zentrum auf u und Achse v transitiv auf den nicht auf u oder v gelegenen Punkten und gilt das Entsprechende für die Gruppe $\mathfrak{P}'_{u',v'}$ einer zweiten Ebene und gibt es einen Isomorphismus φ von $\mathfrak{P}_{u,v}$ auf $\mathfrak{P}'_{u',v'}$, der die Untergruppen $\mathfrak{P}_{Z,v}$ der Kollineationen mit Zentrum Z auf die Untergruppen $\mathfrak{P}'_{Z',v'}$ abbildet, so gibt es einen Isomorphismus σ der ersten Ebene auf die zweite, der einen Punkt $p\notin u, v$ auf einen vorgegebenen Punkt $p'\notin u', v'$ abbildet und der φ induziert. Sind insbesondere die beiden Ebenen gleich und ist φ die Identität, so ist σ eine zentrale Kollineation in $\mathfrak{P}_{u,v}$. (2) Gibt es eine zentrale Dualität mit Zentrum Z und Achse a, so ist die Gruppe $\mathfrak{P}_{Z,a}$ der (Z, a)-Streckungen und die Gruppe $\mathfrak{P}_{Z,Z}$ der Translationen mit Zentrum Z kommutativ. (3) Eine Ebene mit transitiver (Z, a)-Streckungsgruppe, in der es zu zwei verschiedenen Achsen durch Z nicht identische Translationen gibt, ist desarguessch. (4) Eine desarguessche Ebene mit einer quasiprojektiven Dualität ist pappusisch. Dabei heißt eine Dualität δ quasiprojektiv, wenn es für jede

Gerade g mit $g^\delta \notin g$ eine Kollineation α mit $P^\delta \cap g = P^\alpha$ für alle $P \in g$ gibt. *H. Salzmann* (Frankfurt).

See also: General Topology: Viola. **Differential Geometry:** McConnell. **Mechanics of Particles and Systems:** Schatz.

Convex Domains, Integral Geometry

de Groot, J.; and de Vries, H. Convex sets in projective space. Compositio Math. **13** (1958), 113–118.

If any two points of a set S in real projective n-space can be joined by a straight line segment entirely within S, then S is defined to be semi-convex. If, in addition, a hyperplane exists in the projective n-space which does not contain any point of S, then S is defined to be convex. It is shown that a semi-convex set S which does not contain any projective straight line is convex. In the case in which S is assumed to be open, this was proved by Dekker [Amer. Math. Monthly **62** (1955), 430–431; MR **16**, 950]. The authors establish the result first for $n=2$, after which the general result follows easily by induction. *D. Derry* (Vancouver, B.C.).

Kelly, Paul J. Curves with a kind of constant width. Amer. Math. Monthly **64** (1957), 333–336.

If a closed convex curve C in the plane has constant width, its pedal curve $C'=P(C)$ with respect to an arbitrary point A has A as an equichordal point (i.e., all chords of C' through A have the same length). Conversely, given a curve C' star-shaped with respect to A, and with A as an equichordal point, its "negative pedal curve" $C=P^{-1}(C')$ (which is easily obtained as an envelope of straight lines) has constant width provided it is convex. The author considers non-convex curves obtained in this way. Actually, they are inner parallel curves of curves of constant width. [For a rather detailed discussion see Ch. Jordan et R. Fiedler, Contributions à l'étude des courbes convexes fermées, Hermann, Paris, 1912.] *W. Fenchel* (Copenhagen).

Dinghas, Alexander. Über das Verhalten der Entfernung zweier Punktmengen bei gleichzeitiger Symmetrisierung derselben. Arch. Math. **8** (1957), 46–51.

A distance $d(A, B)$ between two point sets A and B in Euclidean n-space E^n is defined by $d(A, B)=L^*[(A-B)\cup(B-A)]$, where L^* denotes outer Lebesgue measure. Let a k-dimensional subspace E^k be given. Symmetrization of an F_σ set $M\subset E^n$ with respect to E^k is defined as follows: Denote by $E_P{}^{n-k}$ the $(n-k)$-flat orthogonal to E^k and passing through the point $P\in E^k$. For each P for which $E_P{}^{n-k}\cap M$ is non-empty, replace this intersection by a solid sphere in $E_P{}^{n-k}$ with center P and having the same $(n-k)$-dimensional Lebesgue measure. The union of all these spheres is the symmetrized set ωM. It is shown that ωM is also F_σ. The following theorem is proved: Let A and B be F_σ sets in E^n. Then $d(\omega A, \omega B)=d(A, B)-2L[\omega(A-B)\cap\omega(B-A)]$. Finally this result is generalized to Riemannian spaces of constant curvature. *W. Fenchel* (Copenhagen).

Dinghas, Alexander. Zum Minkowskischen Integralbegriff abgeschlossener Mengen. Math. Z. **66** (1956), 173–188.

The non-empty, closed and bounded subsets of Euclidean n-space form, with the Fréchet metric, a finitely compact metric space. The author defines a Riemann integral for functions A_λ defined in the interval $J: 0\leq\lambda\leq 1$, with values in this space in the following manner. Let $0=\lambda_0\leq\xi_1\leq\lambda_1\leq\xi_2\leq\lambda_2\leq\cdots\leq\lambda_{l-1}\leq\xi_l\leq\lambda_l=1$, and consider the Minkowski linear combination $\sum_{i=1}^{l}(\lambda_i-\lambda_{i-1})A_{\xi_i}$. If there exists a closed and bounded set S_0 to which this linear combination converges as $\max_i(\lambda_i-\lambda_{i-1})\to 0$, then the function A_λ is called Riemann-Minkowski integrable, and S_0 is written $\int_0^1 d\lambda A_\lambda$. Defining the oscillation $\omega(\lambda)$ of A_λ at $\lambda\in J$ in an obvious manner, the author shows that A_λ is integrable if the sets A_λ, $\lambda\in J$, are convex and uniformly bounded, and the subset $E=\{\lambda|\omega(\lambda)>0\}$ of J has measure 0. For non-convex sets, no general test is given. For a piecewise-constant A_λ, the integral exists and equals a linear combination of the convex hulls $[A_\lambda]$ of the finitely many distinct values of A_λ. In the general case, the author therefore defines $\int_0^1 d\lambda A_\lambda=\int_0^1 d\lambda[A_\lambda]$ provided the latter integral exists. (This shows that the significance of the notion is in the convex case.) The classical Brunn-Minkowski inequality is easily generalized to $L(\int_0^1 d\lambda A_\lambda)^{1/n}\geq\int_0^1 L(A_\lambda)^{1/n}d\lambda$, where L denotes Lebesgue measure. If $L([A_\lambda])>0$ for $\lambda\in J$, equality holds if and only if all the A_λ for λ not in the set E defined above are convex and positively homothetic. The proof of this statement is based on a strengthened Hölder inequality and is rather involved. *W. Fenchel* (Copenhagen).

Hadwiger, H. Über konkave und konvexe Eikörperscharen. Publ. Math. Debrecen **5** (1957), 97–101.

A family K of convex bodies in Euclidean k-space is called a canal family if there exists a $(k-1)$-hyperplane E^0 such that all bodies of K have the same orthogonal projection onto E^0. A one-parameter canal family $K=\{A(\lambda)\}$, where λ runs through some interval J, is called totally concave [convex] if it is concave [convex], that is, $A((1-\theta)\lambda_0+\theta\lambda_1)\supseteq[\subseteq](1-\theta)A(\lambda_0)+\theta A(\lambda_1)$, for $\lambda_0, \lambda_1\in J$, $0\leq\theta\leq 1$, and if the same relation holds for every canal family consisting of the intersections of the bodies $A(\lambda)$ with a flat of dimension $<k$ orthogonal to E^0. Concavity implies total concavity, and total convexity follows from the second condition for straight lines only. Both properties are preserved under orthogonal projections of the family. By means of this remark, it is easily seen that each of the mixed volumes $W_i(A(\lambda))$, $i=0, 1, \cdots, k$, of the bodies $A(\lambda)$ of a totally concave [convex] family is a concave [convex] function of λ. Corollaries: The fact that $A(\lambda)=(1-\lambda)A_0+\lambda A_1$, where A_0 and A_1 have the same projection onto E^0 and $0\leq\lambda\leq 1$, is totally concave, yields linear refinements of the quadratic inequalities between certain mixed volumes of A_0, A_1 and the unit sphere. In the special case where A_1 is the body $\bar{A}_0$ symmetric to A_0 with respect to E^0, the family yields a continuous Blaschke symmetrization, and it follows that the quantities $W_i(A_0)$ do not decrease under this symmetrization. The family

$$A(\lambda)=\mathbf{U}_G[(1-\lambda)A_0\cap G+\lambda A_1\cap G],$$

where G ranges over the lines orthogonal to E^0 and intersecting A_0 and A_1, is totally convex. For $A_1=\bar{A}_0$ it yields a continuous Steiner symmetrization, and it follows that the $W_i(A_0)$ do not increase under this symmetrization. *W. Fenchel* (Copenhagen).

Rogers, C. A.; and Shephard, G. C. The difference body of a convex body. Arch. Math. **8** (1957), 220–233.

Let K be a convex body in Euclidean n-space, and denote by DK its difference (or vector) body $K-K$. The

volumes of these bodies satisfy an inequality of the form $V(DK) \leq c_n V(K)$, where c_n depends only on n. For $n=2$ and 3 the smallest possible c_n have been known for a long time. For $n>3$ the known value of c_n turned out to be far from the best one. Using a lemma due to Th. Bang, the authors prove in an elegant manner that the inequality holds with $c_n=\binom{2n}{2}$, and that this is the best possible value. It is further shown that equality holds only for simplexes. Finally, from the result, a lower estimate, considerably better than those previously known, for the density of packings by the translates of a non-central convex body is deduced. {Reviewer's remark: C. Godbersen, Dissertation, Göttingen, 1938, should be added to the list of references. In this paper, the above results have been conjectured, and proofs are given for particular classes of bodies.} *W. Fenchel* (Copenhagen).

Danzer, Ludwig. Über die maximale Dicke der ebenen Schnitte eines konvexen Körpers. Arch. Math. 8 (1957), 314–316.

The following problem suggested by W. Süss is discussed: Let K be a convex body in ordinary space. Denote by d its minimal width and by d_s the maximum of the minimal widths of the plane sections of K. What is the greatest lower bound of d_s when K ranges over all bodies with a given minimal width d? The author shows that this lower bound is smaller than d by constructing a body of constant width for which $d_s<d$. *W. Fenchel.*

Block, H. D. Discrete isoperimetric-type inequalities. Proc. Amer. Math. Soc. 8 (1957), 860–862.

Given a plane bounded convex domain R of length L and area A, also a square unit lattice. No lattice point shall lie on the boundary of R and no segment of length one bounded by two lattice points shall intersect that boundary twice. Let T denote the number of lattice points in R. We provide each of them with a multiplicity, viz., the number of those of the four nearest lattice points that lie outside of R. Let M denote the sum of all these multiplicities. The author proves: (1) $(M-4)/2^{\frac{1}{2}} \leq L \leq M+4$, (2) $M \geq 4(A^{\frac{1}{2}}-1)$, (3) $M \geq 4T^{\frac{1}{2}}$, (4) $L \geq 8^{\frac{1}{2}}(T^{\frac{1}{2}}-1)$. He also derives estimates for averages if the lattice is rotated. {Reviewer's comments: (i) In (1), (2), and (4) equality should be excluded. (1), (2), (3) are then best possible. (ii) Let N denote the number of lattice points in R with positive multiplicities. The author states that $M \leq 3N$ if $T>1$. This seems to be false. Choose R as $(x+y)^2+15(x-y)^2 \leq 8$. The lattice consists of the points with integral coordinates. Then $T=N=3$, $M=12$.} *P. Scherk.*

Fejes Tóth, L. Über eine Extremaleigenschaft des fünf- und sechseckigen Sternes. Elem. Math. 13 (1958), 32–34.

The regular pentagram $\left\{\frac{5}{2}\right\}$, which is the Pythagorean symbol of health, and the compound of two triangles inscribed in a regular hexagon, which is a sacred sign of the Jews, are extreme cases of a "double oval", which is either a closed curve of non-negative curvature whose total curvature is 4π or a pair of ovals crossing each other at least four times. The part of a double oval between two neighboring double-points is a kind of "lune" consisting of two arcs. Consider the difference between the lengths of these two arcs. Let λ be the minimum difference for all the lunes that occur in a given double oval of total length Λ having $n>4$ double-points. The author proves that

$$\frac{\Lambda}{\lambda} \geq n \cot^2 \frac{\pi}{n},$$

with equality only for an n-pointed star. He shows how one might extend his method to prove a somewhat similar inequality for areas, conjectured by Erdös.

H. S. M. Coxeter (Toronto, Ont.).

Volland, Walter. Ein Fortsetzungssatz für additive Eipolyederfunktionale im euklidischen Raum. Arch. Math. 8 (1957), 144–149.

Let M be the class of polyhedrons in k-dimensional euclidean space E_k, and M_0 the sub-class of convex polyhedrons (the null set Λ included). Let G be an additive abelian group with the null element 0 and let φ_0 be a single-valued map of M_0 into G such that: (a) $\varphi_0(\Lambda)=0$; (b) $\varphi_0(A \cup B)+\varphi_0(A \cap B)=\varphi_0(A)+\varphi_0(B)$ $(A, B, A \cup B \in M_0)$. Then φ_0 can be extended in one and only one way to a map φ of M into G which preserves the properties (a), (b), with $A, B \in M$ and $\varphi(A)=\varphi_0(A)$ for $A \in M_0$.

L. A. Santaló (Buenos Aires).

See also: Topological Vector Spaces: Klee. **Differential Geometry:** Valette.

Differential Geometry

★ **McConnell, A. J. Application of tensor analysis.** Dover Publications, Inc., New York, 1957. xii+318 pp. $1.85.

A reprint, unchanged except for the title, of the author's well-known book first published by Blackie in 1931 under the title "Applications of the absolute differential calculus."

Scherrer, W. Die Grundgleichungen der Flächentheorie. II. Comment. Math. Helv. 32 (1957), 73–84.

This is a continuation of the first paper of the same title [same Comment. 29 (1955), 180–198; MR 16, 1050]. Here the main object is to find the relation between the mean curvature and the "principal" angle ω ($\omega/2$ is the angle subtended by a principal direction and the bisector of the angle of the net curves). The computation is not invariant. *V. Hlavatý* (Bloomington, Ind.).

Kaul, R. N. Generalized normal curvature of a vector field. Tensor (N.S.) 7 (1957), 110–116.

Given any vector field on a surface in a Euclidean space of three dimensions, notions of associate curvature and normal curve of such a field with respect to a curve have been introduced by Graustein [Trans. Amer. Math. Soc. 34 (1932), 557–593] and by Pan [Amer. J. Math. 74 (1952), 955–966; MR 14, 406] respectively. In this paper a generalization of the notion of normal curvature of a vector field is given. The corresponding notion is then given for a subspace V_n of a Riemannian space V_{n+1}. The possible definitions of notions corresponding to those of asymptotic lines and principal directions are given. *E. T. Davies.*

Vincensini, Paul. Sur la déformation de certains réseaux isogonaux d'une surface. Boll. Un. Mat. Ital. (3) 12 (1957), 377–384.

Let Σ denote a sphere with center at any point $P(u, v)$ of an arbitrary surface S of ordinary Euclidean space, and with radius $R(u, v)$. Such a sphere Σ touches the envelope of this congruence of spheres (Σ) in two points M_1 and

M_2, which are symmetrical relative to the tangent plane π of S at the point P. The midpoint I of the line segment M_1M_2 is in the plane π. The author develops some results concerning this law of association (P, I). If as the point P describes a curve C on S, the point I describes a curve C' such that C and C' are instantaneously orthogonal, then all such curves C form a net N on the surface S. The important case is considered when the net N is isogonal, that is, at any point P of S, the corresponding two directions of the net N form the constant angle α. Under these conditions, the point I is on the focal segment F_1F_2, where F_1 and F_2 are the two centers of principal curvature of S at P, and divides it in the constant ratio $k=-\tan^2\alpha/2$. Also, the set of all planes each one of which is orthogonal to a focal segment F_1F_2 at the point I is tangent to a sphere. Finally, when N is an orthogonal net, applications are made to a developable surface S and to a surface S applicable to a sphere. *J. De Cicco* (Chicago, Ill.).

Švec, Alois. Sulla teoria delle congruenze di rette. Boll. Un. Mat. Ital. (3) **12** (1957), 446–457.

Relazione, senza dimostrazioni, dei risultati ottenuti in Cecoslovacchia sulla geometria differenziale (locale) delle congruenze di rette. — I principali argomenti toccati sono: congruenze a connessione proiettive, che si definiscono come segue: sia $\sigma(u, v)$ un dominio di uno spazio euclideo; se $P \in \sigma$, si fa corrispondere a P uno S_3 proiettivo, e in questo una retta $p(u, v)$; ad ogni arco congiungente P_1, $P_2 \in \sigma$ si associa una omografia fra gli spazi $S_3(P_1)$, $S_3(P_2)$, dipendente dall'arco; la connessione è individuata dalle omografie locali fra $S_3(u, v)$ ed $S_3(u+du, v+dv)$, introducendo in ogni S_3 un riferimento mobile A_1, A_2, A_3, A_4, di guisa che p passi per A_1, A_2. — Per tali connessioni si dànno molti teoremi analoghi a quelli sulle congruenze di rette in S_3. Si studiano poi anche le congruenze considerate immerse nello spazio rigato a quattro dimensioni. — Si enunciano poi vari risultati, dovuti in parte a E. Čech, sulle congruenze di rette immerse in uno spazio proiettivo, e sulle loro deformazioni proiettive. *V. Dalla Volta* (Roma).

Süss, Wilhelm. Eindeutige Bestimmung von Eihyperflächen durch die Summe ihrer Hauptkrümmungsradien. Arch. Math. **8** (1957), 352–354.

Kurz und elegant Beweis der folgende Satz (bewiesen von Christoffel für $n=2$): Die beiden Hyperflächen x und $\bar{x}$ des $(n+1)$-dimensionalen euklidischen Raumes seien geschlossen und durch parallele und gleichgerichtete Normalen eineindeutig sphärisch, also auch aufeinander abgebildet. Ihre Hauptkrümmungsradien seien positiv und stetig; die Summen S bzw. $\bar{S}$ dieser Radien R_i bzw. $\bar{R}_i$ ($i=1, 2, 3, \cdots, n$) mögen als Funktionen des Ortes auf dem gemeinsamen sphärischen Bild von x und $\bar{x}$ miteinander übereinstimmen: $S=\sum R_i=\bar{S}=\sum \bar{R}_i$. Dann sind die beiden Hyperflächen x und $\bar{x}$ einander translationsgleich. *L. A. Santaló* (Buenos Aires).

Su, Buchin. Godeaux sequences and associate Laplace sequences of a projective minimal surface. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 569–576.

L'auteur appelle transformée D d'une surface la surface engendrée par un quelconque des quatre sommets du quadrilatère de Demoulin de la première. Les asymptotiques se correspondent sur une surface minimale projective S et sur ses transformées D. Si $\bar{S}$ est l'une de celles-ci, S est aussi transformée D de $\bar{S}$. Il existe deux congruences W qui transforment S en une transformée D de $\bar{S}$; les images dans l'espace à 5 dimensions des rayons de ces congruences forment deux suites de Laplace (W) et $(\bar{W})$ qui peuvent être associées de façon que les rayons correspondants de ces suites se coupent, déterminant la suite de Godeaux conjuguée simultanément à (W) et $(\bar{W})$. *M. Decuyper* (Lille).

Speranza, Francesco. Proprietà proiettive delle trasformazioni dualistiche. Boll. Un. Mat. Ital. (3) **12** (1957) 552–565.

L'Autore studia dal punto di vista della geometria proiettiva una trasformazione dualistica T (non di tipo nullo) tra due piani sovrapposti π, $\bar{\pi}$, introducendo nuovi enti proiettivamente covarianti: in particolare le "curve canoniche" di T, così definite. Se A, a sono un punto di π ed una retta di $\bar{\pi}$ omologhi in T e non appartenentisi, resta subordinata tra le direzioni uscenti da A ed i punti di a una proiettività, e quindi anche — proiettando a da A — una proiettività Γ (introdotta da E. Bortolotti) tra due fasci sovrapposti di sostegno A. Le curve canoniche di T sono quelle curve che in ogni loro punto sono tangenti ad una delle direzioni unite in Γ. In modo duale si possono introdurre gli "inviluppi canonici". L'Autore studia le trasformazioni T in relazione agli enti definiti e ad altri già noti. *D. Gallarati* (Genova).

Vranceanu, Gheorghe. Sul tensore associato ad una corrispondenza fra spazi proiettivi. Boll. Un. Mat. Ital. (3) **12** (1957), 489–506.

Siano $A_n(x^1, \cdots, x^n)$, $E_n(u^1, \cdots, u^n)$ due spazi proiettivi, e $u^i=u^i(x^1, \cdots, x^n)$ una corrispondenza puntuale fra essi. Si può allora definire in A_n un tensore $\Pi_{jh}{}^i$ — associato alla corrispondenza $u \leftrightarrow x$ — le cui componenti non sono altre che le componenti della connessione proiettiva dello spazio a connessione affine euclidea E_n, per le quali le u^i sono coordinate cartesiane [Vranceanu, lo stesso Boll. (3) **12** (1957), 145–153; MR **19**, 764]. Nella presente Nota, l'A. cerca anzitutto le condizioni affinché un dato tensore $\Pi_{jh}{}^i$ — tensore rispetto a trasformazioni proiettive — possa essere associato ad una corrispondenza: trova così delle condizioni differenziali, soddisfatte le quali la trasformazione è definita a meno di una proiettività. — Si studiano poi — specie nel caso $n=2$ — alcune forme definite da $\Pi_{jh}{}^i$, mediante le quali si caratterizzano tipi speciali di trasformazioni. Si fanno poi applicazioni alla classificazione delle trasformazioni puntuali, e ai gruppi di trasformazioni in sè di una corrispondenza. *V. Dalla Volta* (Roma).

Valette, Guy. Quelques propriétés conformes globales des courbes planes. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 66–79.

The notion of vertex of a plane curve belongs to inversive (Moebius) geometry. A vertex of a (real) analytic curve is defined as a point at which the curve and the osculating circle have contact of odd order. Using exclusively notions and tools belonging to inversive geometry, the author proves various, partly known, global properties of simple plane curves and arcs related to the Four-Vertex Theorem. The following result seems to be new: Let c be a simple arc without vertex, and let H and K denote the two null circles of the pencil of circles determined by the osculating circles at the end-points of c. Then the circular arc HPK turns through an angle greater than π about H when the point P traverses c. The Four-Vertex Theorem is proved for arbitrary simple closed curves. From the Moebius invariance it follows

immediately that it also holds in hyperbolic and spherical geometry, and further in elliptic geometry for curves homologous to zero, while the minimum number of vertices of curves non-homologous to zero is shown to be three. {Several relevant papers are not mentioned; in particular, D. Fog, S.-B. Preuss. Akad. Wiss. **1933**, 251–254, where the Four-Vertex Theorem is proved under even weaker assumptions and where its inversive character is emphasized.} *W. Fenchel* (Copenhagen).

Manifolds, Connections

Stojanović, Rastko. Motion of a rigid body in two dimensional Riemannian space. Bull. Acad. Serbe Sci. Cl. Sci. Math. Nat. (N.S.) **10** (1956), no. 2, 43–47.

The author considers the number of degrees of freedom of a rigid body in a two dimensional Riemannian space. This depends on the nature of the fundamental form $ds^2=\sum g_{ij}dx^idx^j$. He shows that the number of degrees of freedom, i.e., the number of parameters required to specify the position of the body, is one only if $ds^2=(dx^1)^2+f(x^1)(dx^2)^2$, and is two or three only if the metric is flat or of constant Gaussian curvature. He concludes with a derivation of Lagrangian equations for this general case. *W. M. Boothby* (Evanston, Ill.).

Debever, R. Connexions métriques et champs d'éléments plans parallèles dans les variétés à quatre dimensions. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 56–61.

It is shown that in a Riemannian 4-space with a given field of 2-planes there are connexions which make the given planes parallel and are metric in the sense that $\nabla_k g_{ij}=0$ where g_{ij} is the metric tensor. Locally these connexions depend upon two arbitrary vector fields, and particular forms are considered. *A. G. Walker.*

Gasparini Cattaneo, Ida. Sulle geodetiche di una V_n relative a una connessione affine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 146–154.

According to A. Lichnerowicz [Théorie globale des connexions..., Edizioni Cremonese, Roma, 1957; MR **19**, 453], a linear connection on a differentiable V_n is an infinitesimal connection on the principal fiber bundle $E(V_n)$ of the frames in V_n. An affine connection is also, an infinitesimal connection, but on the fiber bundle $\mathfrak{E}(V_n)$ of the affine frames (an affine frame at $x\in V_n$ being the collection of a vector $\lambda\in V_n$ and a basis in V_n). In a previous paper, the author [Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 119–126; MR **19**, 453] had studied the affine connections associated to a given linear connection. Here, she continues her research, and determines the differential equations of the geodesics of an affine connection, i.e., the lines which are developed in straight lines. For these geodesics the differential — according to the given affine connection π — of the vector $\overset{\pi}{\nabla}x/dt$ is collinear to $\overset{\pi}{\nabla}x/dt$. The latter, however, is not the tangent vector to the geodesic, but is obtained by it through a linear mapping defining the connection π with respect to a given linear connection; if the affine connection is the canonical one, its geodesics coincide with the geodesics of the linear connection, but the converse is not true. — Some applications are made to the theory of general relativity. *V. Dalla Volta* (Rome).

Vranceanu, G. Sur les groupes de mouvements des espaces à connexion affine projectifs. Ann. Sci. Ecole Norm. Sup. (3) **74** (1957), 249–268.

Il presente lavoro è diviso in due parti principali: in entrambe si studiano spazi a connessione affine proiettivamente euclidei, per i quali cioè esiste un sistema globale di coordinate x^i ($i=1,\cdots,n$) nel quale le autoparallele siano delle rette: in un tale sistema le componenti della connessione, supposta simmetrica sono date da $\Gamma_{hk}^i=\delta_h^i\varphi_k+\delta_k^i\varphi_h$. Nella p. I l'A. studia la possibilità di estendere la connessione a tutto lo spazio, e dimostra che ciò è senz'altro possibile se le funzioni φ_i sono analitiche intere: si dànno anche esempi in cui tale estensione invece non è possibile. — Nella p. II, si studiano i gruppi di movimenti per uno spazio proiettivamente euclideo, ma non localmente euclideo, in relazione all'esistenza o meno di pfaffiani invarianti, ritrovando, fra l'altro, e con metodo più diretto, risultati di Egorov; precisamente, l'A. ottiene i suoi risultati direttamente dalle formule di trasformazione dei parametri Γ_{hk}^i della connessione. In vari casi si determina l'ordine massimo del gruppo di movimenti, e si dànno esempi di spazi possedenti effettivamente un gruppo dell'ordine considerato. *V. Dalla Volta* (Roma).

Ishihara, Shigeru; and Fukami, Tetsuzo. Groups of affine transformations and groups of projective transformations in a space of K-spreads. Jap. J. Math. **26** (1956), 79–93.

This paper is concerned with the spaces of K-spreads which admit groups of affine or projective transformations of sufficiently high dimension. The following are proved: (I) Let G be a group of affine transformations in an N-dimensional space of K-spreads with dim $G>N^2$. Then G is transitive, the space is affinely flat, and dim $G=N^2+N$, N^2+N-1 or N^2+1. Moreover, the space is homeomorphic with either a euclidean space or the product of a circle and a euclidean space. (II) Let G be a group of projective transformations in an N-dimensional space of K-spreads. If dim $G\geqq N^2+N$, then the space is projectively flat, and dim $G=N^2+2N$ or N^2+N. *H. C. Wang.*

See also: Partial Differential Equations: Calabi. **Lie Groups and Algebras:** Wallace. **Differential Geometry:** Kaul; Vranceanu.

Complex Manifolds

Grauert, Hans; et Remmert, Reinhold. Faisceaux analytiques cohérents sur le produit d'un espace analytique et d'un espace projectif. C. R. Acad. Sci. Paris **245** (1957), 819–822.

Cette Note annonce, sans démonstration, les trois théorèmes que voici. Soient Y un espace analytique, et P_n l'espace projectif complexe de dimension n; soit $\tau: Y\times P_n\to Y$ la première projection. Soit F le fibré canonique de base P_n, à fibres vectorielles de dimension 1 (F se compose des points $(u,v)\in P_n\times C^{n+1}$ tels que la droite de C^{n+1} définie par u passe par v), et soit $\mathfrak{F}$ le faisceau, sur $Y\times P_n$, formé des germes de sections holomorphes de $Y\times F$ (fibré de base $Y\times P_n$). Pour chaque entier $k\geqq 0$, soit $\mathfrak{F}^k$ le produit tensoriel (pris sur le faisceau d'anneaux $\mathfrak{O}$ des germes de fonctions holomorphes sur $Y\times P_n$) de k exemplaires de $\mathfrak{F}$. Dans tout ce qui suit, $\mathfrak{S}$ désigne un faisceau analytique cohérent sur $Y\times P_n$, et $\mathfrak{S}^k$ désigne le produit tensoriel $\mathfrak{S}\otimes_{\mathfrak{O}}\mathfrak{F}^k$.

Th. 1: Pour tout $x\in Y\times P_n$ existe un entier $k_0(x,\mathfrak{S})$

tel que, pour tout $k \geq k_0(x, \mathscr{S})$: 1) $H^0(\tau^{-1}(\tau(x)), \mathscr{S}^k)$ engendre $(\mathscr{S}^k)_x$ comme $\mathscr{O}_x$-module; 2) $H^q(\tau^{-1}(\tau(x)), \mathscr{S}^k)=0$ pour $q \geq 1$. De plus, $k_0(x, \mathscr{S})$ est uniformément borné quand x parcourt un compact de $Y \times P_n$.

Th. 3: Si Q est un ouvert relativement compact de Y, et si Q est holomorphiquement complet, il existe $k_0(Q, \mathscr{S})$ tel que, pour tout $k \geq k_0(Q, \mathscr{S})$: 1) $H^0(Q \times P_n, \mathscr{S}^k)$ engendre $(\mathscr{S}^k)_x$ en tout point $x \in Q \times P_n$; 2) $H^q(Q \times P_n, \mathscr{S}^k)=0$ pour $q \geq 1$.

Le th. 2 fait intervenir la notion d'image directe d'un faisceau. Soit $f: X \to Y$ une application continue d'espaces topologiques, et soit $\mathscr{A}$ un faisceau sur X; à tout ouvert U de Y associons $H^0(f^{-1}(U), \mathscr{A})$, ce qui définit un préfaisceau sur Y; le faisceau associé se note $f(\mathscr{A})$; $f(\mathscr{A})$ est un foncteur covariant de $\mathscr{A}$, exact à gauche, et son q-ième foncteur dérivé $R^q f(\mathscr{A})$, noté $f_q(\mathscr{A})$, est le faisceau associé au préfaisceau qui, à chaque ouvert $U \subset Y$, fait correspondre $H^q(f^{-1}(U), \mathscr{A})$. Th. 2: avec les notations ci-dessus, le faisceau $\tau_q(\mathscr{S})=R^q\tau(\mathscr{S})$ est un faisceau analytique cohérent sur Y. *H. Cartan.*

Grauert, Hans; et Remmert, Reinhold. Espaces analytiquement complets. C. R. Acad. Sci. Paris **245** (1957), 882–885.

An analytic space X is analytically complete if it is holomorphically convex and if, for each point $x_0 \in X$, there is a mapping f of X into a complex projective space such that x_0 is an isolated point of the set $\{x | f(x)=f(x_0)\}$. The authors list several results which they have obtained for this class of spaces. For example, every analytic polyhedron in such a space is equivalent to an analytic subvariety of a product of a polycylinder and a projective space. Further theorems generalize results obtained in their previous note [reviewed above], and the imbedding theorem of Chow [Amer. J. Math. **71** (1949), 893–914; MR **11**, 389]. To describe the final result, let $f: X \to Y$ be a proper analytic mapping of such a space X onto a connected analytically complete space Y, and let S be a coherent analytic sheaf over X. The groups $H^q(X, S)$ can be considered as modules over the ring of holomorphic functions on Y under the map f; let n_q be the ranks of these modules, and $\chi_f(X, S)=\sum (-1)^q n_q$. Then for all points $y \in Y$ outside of an analytic subvariety of Y, $\chi_f(X, S)=\chi_f(f^{-1}(y), S)$. *R. C. Gunning.*

Grauert, Hans; et Remmert, Reinhold. Sur les revêtements analytiques des variétés analytiques. C. R. Acad. Sci. Paris **245** (1957), 918–921.

H. Cartan defined a complex space with singularities by means of local elements equivalent to normal analytic subvarieties of a complex manifold [Séminaire H. Cartan Ecole Norm. Sup., 1951–52, Exposé XIII; MR **16**, 233]; H. Behnke and K. Stein defined a complex space with singularities by means of local elements equivalent to branched analytic coverings of a complex manifold [Math. Ann. **124** (1951), 1–16; MR **13**, 644]. It is clear that the class of spaces defined by Cartan is contained in the class of spaces defined by Behnke-Stein; the present note briefly outlines a proof that, conversely, the class of spaces defined by Behnke-Stein is contained in the class of spaces defined by Cartan; hence, that the two definitions are equivalent. The problem is, of course, purely local, so that it is enough to show that every branched covering $\tilde{X}$ of a complex manifold X can locally be imbedded as a normal analytic subvariety of a complex manifold; and this in turn is reduced to showing that the ring of germs of analytic functions at a point of $\tilde{X}$, where $\tilde{X}$ is a b-sheeted covering, is an integral algebraic extension of degree b of the subring of functions holomorphic on X. The authors reduce this to the spcieal case in which $X=G \times P_1$, where G is a complex manifold and P_1 is one-dimensional complex projective space (the Riemann sphere); then the techniques used by Serre on projective spaces [Séminaire H. Cartan Ecole Norm. Sup., 1953-1954, Exposées XVIII–XX; MR **19**, 577] and developed in the paper reviewed above are applied. *R. C. Gunning.*

Hirzebruch, F.; and Kodaira, K. On the complex projective spaces. J. Math. Pures Appl. (9) **36** (1957), 201–216.

Let X be an n-dimensional compact Kähler manifold which is C^∞-differentiably homeomorphic to the complex projective space $P_n(\mathbf{C})$ with its usual differentiable structure. The main result of the present paper is that, if the dimension n of X is odd, then X is complex-analytically homeomorphic to $P_n(\mathbf{C})$, i.e., there exists a bi-regular map of X onto $P_n(\mathbf{C})$. This result had been earlier announced in two papers of Hirzebruch [Ann. of Math. (2) **60** (1954), 213–236, problem 28*); Proc. Internat. Congress Math. 1954, Amsterdam, vol. III, Noordhoff, Groningen, 1956, pp. 457–473; see pp. 468–469; MR **16**, 518; **19**, 317].

The main theorem is proved by the authors in two steps. The first step is expressed by the following characterization of the complex projective spaces $P_n(\mathbf{C})$ of any dimension n. Let X be an algebraic variety of dimension n having the following properties: (i) the cohomology ring of X has the form

$$H^*(X, \mathbf{Z})=\mathbf{Z}+\mathbf{Z}g+\mathbf{Z}g^2+\cdots+\mathbf{Z}g^n,$$

where the generator g belongs to $H^2(X, \mathbf{Z})$; (ii) there exists a Kähler metric on X whose fundamental class coincides with the generator g; (iii) there exists a holomorphic divisor D on X whose cohomology class $c(D)$ is equal to g; (iv) for any natural integer ν the dimension of the space $H^0(X, \nu D)$ is given by $\dim H^0(X, \nu D)=\binom{n+\nu}{n}$. Then X is complex-analytically homeomorphic to the complex projective space $P_n(\mathbf{C})$. This theorem is proved by showing that, if $\varphi_{0j}, \cdots, \varphi_{nj}$ are the local representations of a base of $H^0(X, D)$, then the functions $f_{ij}=\varphi_{ij}/\varphi_{0j}$ — which are clearly local representations of rational functions on X — define a bi-regular transformation (i.e. a complex analytic homeomorphism) $z \to (1, f_1(z), \cdots, f_n(z))$ $(z \in X)$ of X onto $P_n(\mathbf{C})$. By using the general Riemann-Roch theorem [Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Springer, Berlin-Göttingen-Heidelberg, 1956; MR **18**, 509] and a theorem on vanishing of cohomology groups [K. Kodaira, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1268–1273; MR **16**, 618], the authors prove the following proposition, which constitutes the second step of the proof of the above stated main theorem. Let X be a compact Kähler manifold which is differentiably homeomorphic to the complex projective space $P_n(\mathbf{C})$. Then X is an algebraic manifold. Let g be the generator of $H^2(X, \mathbf{Z})$ whose sign is chosen in such a way that g is represented by the fundamental class of one of the Kähler metrics on X. Let $D \to c(D)$ be the mapping which associates to every divisor D the characteristic class $c(D) \in H^2(X, \mathbf{Z})$ of the complex line bundle (determined up to an isomorphism) belonging to (the divisor class of) D. This mapping induces an isomorphism of the group of the divisor classes of X onto $H^2(X, \mathbf{Z})$. Then, if n is odd, the first Chern class $c_1 \in H^2(X, \mathbf{Z})$ of X is expressed by (*) $c_1=(n+1)g$, while, if n is even,

$c_1=\pm(n+1)g$. If (*) holds, then for every divisor D, with $c(D)=\nu g$ ($\nu\in\mathbf{Z}$, $\nu>0$), we have that dim $H^0(X,\nu D)=\binom{n+\nu}{n}$ while, if $c_1=-(n+1)g$, and if $\nu>n+1$, then dim $H^0(X,\nu D)=\binom{\nu-1}{n}$.

It follows from the general Riemann-Roch theorem and from Kodaira's vanishing theorems that, if (n is even and if) $c_1=-(n+1)g$, the plurigenera $P^r=\dim H^0(X,rK)$ ($r=1, 2, \cdots$) are expressed by $P^r=\binom{nr+r-1}{n}$. But the plurigenera of $P_n(\mathbf{C})$ are all zero. Furthermore they are birational invariants [B. L. van der Waerden, Math. Z. **51** (1948), 502–523; MR **10**, 622]. Thus if the dimension n of X is even the conclusion of the main theorem can be replaced, for example, by: "If the bigenus P^2 is different from $\binom{2n+1}{n}$, then X is complex-analytically homeomorphic to $P_n(\mathbf{C})$". It is not known whether the case n even and $c_1=-(n+1)g$ actually occurs. The Pontrjagin classes occur in the general Riemann-Roch theorem. This is the reason why the homeomorphism between X and $P_n(\mathbf{C})$ iss upposed to be differentiable. But it is unknown whether the main theorem remains correct if "differentiably homeomorphic" is replaced by "homeomorphic". Since for $n=1$ no Pontrjagin classes occur, and since the Pontrjagin class of a 4-dimensional manifold is a topological invariant, the main theorem is proved for $n<3$ under this weaker assumption. The authors remark that nothing is known about non-Kähler complex manifolds of dimension $n>1$ which are differentiably homeomorphic to $P_n(\mathbf{C})$. *E. Vesentini* (Evanston, Ill.).

Algebraic Geometry

★ **Weil, André. Abstract versus classical algebraic geometry.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 550–558. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Die "klassischen" und die "abstrakten" Methoden in der algebraischen Geometrie werden ihrem Wesen und ihrer Reichweite nach miteinander verglichen. Zur Erläuterung dienen mehrere Beispiele: Castelnuovos Satz über den Äquivalenzdefekt einer Korrespondenz zweier Kurven (für den ein einfacher Beweis im klassischen Fall gegeben wird); der Zusammenhang dieses Satzes mit der Riemannschen Vermutung in Funktionenkörpern; Verallgemeinerung der Riemannschen Vermutung für höhere Dimension; Lefschetzs Fixpunktformel und der Zusammenhang mit den "arithmetischen" Zetafunktionen.

P. Roquette (Hamburg).

Godeaux, Lucien. Observation sur la construction de surfaces algébriques non rationnelles de genres zéro. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 587–589.

Godeaux, Lucien. Sur la construction de surfaces projectivement canoniques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 699–704.

Godeaux, Lucien. Note sur les points de diramation d'une surface multiple. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 8–16.

Godeaux, Lucien. Involutions rationnelles ayant quatre points unis appartenant à une surface algébrique non rationnelle. Bull. Soc. Roy. Sci. Liège **26** (1957), 388–389.

Chisini, O. Teoria elementare delle cubiche piane. Period. Mat. (4) **35** (1957), 125–148.

Verf. ist bestrebt zu zeigen daß die klassischen Methoden der algebraischen Geometrie, insbesondere der italienischen geometrischen Schule, und zwar wenige und einfache ihrer Gesichtspunkte, die wichtigsten Eigenschaften der ebenen kubischen Kurven in suggestiver, elementarer und eleganter Weise zu liefern imstande sind. Er geht aus von dem Fundamentalsatz, daß alle kubischen Kurven, die durch acht Schnittpunkte von zwei gegebenen kubischen Kurven, die weder einen Bestandteil noch eine Dopplpunkt gemein haben, auch durch den neunten gehen. Daraus folgt sofort der Satz von Pascal über Kegelschnitte, der Satz über die neun Schnittpunkte einer kubischen Kurve mit drei Geraden, der Satz von Poncelet, daß drei Geraden durch einen Wendepunkt, und nur Geraden durch einen solchen, die kubische Kurve noch in sechs Punkten eines Kegelschnittes schneiden, die Projektivitäten der kubischen Kurven in sich, die Sätze über die Wendepunktskonfiguration, ferner der Satz von Salmon, daß die vier Tangenten von einem veränderlichen Punkt einer kubischen Kurve an diese ein festes Doppelverhältnis haben, und daß Kurven, für die diese Doppelverhältnisse einander gleich sind, sich projektiv in einander transformieren lassen. Die Arbeit dürfte auch von einem Leser, der der algebraischen Geometrie ferner steht, leicht und mit Genuß zu lesen sein.

O.-H. Keller (Halle).

Wallace, Andrew H. Tangency properties of algebraic varieties. Proc. London Math. Soc. (3) **7** (1957), 549–567.

In this paper the author continues his study of the relationship between a variety V and its dual V' when the ground field is arbitrary [same Proc. (3) **6** (1956), 321–342; MR **18**, 234].

If k is the field of complex numbers then, for all V, duality is symmetric and the contact locus $C(x)$ of the tangent space $T(x)$ to V at a generic point (x) is a linear variety which corresponds by duality to a tangent space of V'. If k is arbitrary and these properties hold, then the classical case of duality is said to hold for V.

Let V be an affine hypersurface defined over k, and let $T(x)$ be the tangent space to V at a generic point (x); then a "characteristic linear variety" $\chi(x)$ can be defined in $T(x)$ such that $\chi(x)$ lies on V if, and only if, the classical case of duality holds for V. $\chi(x)$ always contains the contact locus $C(x)$ of $T(x)$ on V, and the classical case of duality holds only when $\chi(x)$ and $C(x)$ coincide.

The characteristic linear varieties of V arise as the generic tangent spaces of the subvarieties of V which correspond by duality to generic linear varieties (of the right dimension) in the ambient of V'. *D. Kirby.*

Chisini, Oscar. Schemi e modelli per le singolarità delle curve algebriche piane. Confer. Sem. Mat. Univ. Bari no. **23** (1957), 27 pp.

Using the "tree" representation developed by Enriques, the author analyzes compound singularities of algebraic plane curves. In this schematic representation, from the final position O of the singularity branches go out in the ultimate directions of tangents along which

points P_i of given multiplicities have moved into coincidence at O to form the compound singularity. Likewise each point P_i may have branches along which points Q_j of given multiplicities have moved up to that P_i. Also each point Q_j may have similar branches, etc. In accordance with the results of two earlier papers by the author, he often defines a superlinear branch of an algebraic plane curve by a product of substitutions. The number of intersections that occur at a common point O of two algebraic plane curves when each has a given singularity at O is determined; and also the reduction in the class of a curve due to a given singularity. *T. R. Hollcroft.*

See also: Fields, Rings: Yamanashi. **Geometries, Euclidean and Other:** Rosati. **Complex Manifolds:** Hirzebruch and Kodaira.

NUMERICAL ANALYSIS

Numerical Methods

Booth, Andrew D. The efficiency of certain methods of information retrieval. Information and Control **1** (1958), 159–164.

This paper describes the standard methods of table look-up; a method is shown for reducing the number of entries to the table. Unfortunately there are a large number of small errors in the paper, and the formulae quoted are wrong in some cases. *C. B. Haselgrove.*

Sobol, I. M. Multidimensional integrals and the Monte-Carlo method. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 706–709. (Russian)

Value of the integral $I=\int_K f(P)dV$ over a unit cube K of dimension d is approximated by the arithmetic mean

$$I=N^{-1}\sum_{\mu=1}^{N} f(P_\mu)$$

with error Δ_N. The author is concerned with estimates of Δ_N, and compares it with the error when the Monte Carlo method is used. The article is very sketchy, and no proofs are given. *R. E. Gaskell* (Seattle, Wash.).

Greville, T. N. E. On smoothing a finite table: A matrix approach. J. Soc. Indust. Appl. Math. **5** (1957), 137–154.

Matrix algebra provides an approach to a smoothing process which occurs in smoothing a finite table by a "weighted moving average" formula in order to extend to the data near the ends of the table. A type of criterion is suggested here for the determination of the "best" smoothing process which reproduces without change all sequences of a designated class. In the formulation of such criteria, use is made of a "metric" designed to reflect the appropriate weighting of the different observations and a projector which resolves a vector into "smooth" and "nonsmooth" components. *E. Frank.*

Kelley, James E., Jr. An application of linear programming to curve fitting. J. Soc. Indust. Appl. Math. **6** (1958), 15–22.

Let the set $\{x_i: 1\leqq i\leqq m\}$ of real numbers and the values of the real functions f and $\{g_j: 0\leqq j\leqq n\}$ on this set be given. Let A denote the set of all real n-tuples $\alpha=(\alpha_0, \cdots, \alpha_n)$. The "curve-fitting" problem $\mathrm{Min}_{\alpha\in A}\,\mathrm{Max}_{1\leqq i\leqq m}[f(x_i)-\sum_j \alpha_j g_j(x_i)]$ is reformulated as the linear programming problem $\mathrm{Min}\{\lambda: \lambda\geqq\pm(f(x_i)-\sum_j \alpha_j g_j(x_i)),\ 1\leqq i\leqq m\}$, the dual of which is:

$$\mathrm{Min}\{w: w+\textstyle\sum_i (u_i-v_i)f(x_i)=0,\ \sum_i (u_i+v_i)=1,$$
$$\textstyle\sum_i (u_i-v_i)g_j(x_i)=0,\ u, v\geqq 0,\ 0\leqq j\leqq n\}.$$

This formulation is particularly efficient computationally for $m>n$, and has the property that the solution for $\{g_0, \cdots, g_n\}$ is readily obtained from that for $\{g_0, \cdots, g_{n-1}\}$ A specialized algorithm based on this property is given for the case $g_j(x)=x^j$. *P. Wolfe* (Santa Monica, Calif.).

Forsythe, George E. Generation and use of orthogonal polynomials for data-fitting with a digital computer. J. Soc. Indust. Appl. Math. **5** (1957), 74–88.

The goal of the paper is to develop methods for curve fitting by polynomials, and to adapt them to automatic computation. The basic principle is least squares approximation. It is shown that the straight-forward method of Gaussian normal equations leads to a very ill-conditioned system of linear equations, the matrix being asymptotically the famous Hilbert matrix. Therefore, the use of orthogonal polynomials is recommended, and the corresponding theory outlined. A very efficient procedure for generating orthogonal polynomials in the computer is described, using the well known three-term recurrence. Finally, the author gives explicit programs for automatic curve fitting. *E. Stiefel* (Zürich).

Marek, Jindřich M. Interpolation of cotg α in the neighbourhood of $\alpha=0$. Stroje na Zpracování Informací 3 (1955), 197–210 (1956). (Czech. Russian and English summaries)

A method is given for the determination of the value of cot α for small angles of α which is suitable for punched-card computations, since it obviates division. It consists of the introduction of a correction function k_n so that

$$\cot\alpha=n\cot n\alpha+k_n.$$

The function k_n is tabulated for $n=10$ and 100, to be used in conjunction with six-place tables with argument intervals $h=\pi/20000$. Linear interpolation is then used. *V. Vand* (University Park, Pa.).

Stojaković, Mirko. Solution du problème d'inversion d'une classe importante de matrices. C. R. Acad. Sci. Paris **246** (1958), 1133–1135.

The fundamental problem of interpolation, that is, to determine the polynomial $\sum_{\nu=0}^{n} a_\nu x^\nu$ of degree $\leqq n$ taking prescribed values y_k at the points x_k $(k=0, 1, \cdots, n)$, may be formulated as a system of linear equations $\sum_{\nu=0}^{n} a_\nu x_k^\nu=y_k$ in the unknown coefficients a_ν with the Vandermonde matrix $V_n=(x_\mu^\nu)$. The author gives (without proof) a representation of V_n^{-1} in the form $S_nV_n'L_n$, where S_n is an $(n+1)\times(n+1)$-symmetric matrix, the elements $s_{\nu\nu'}$ of which are integral symmetric functions of $x_0, \cdots, x_n$, $s_{\nu\nu'}=0$ for $\nu+\nu'>n$, and L_n is a diagonal matrix, with the elements $L_j=\prod_{\nu=0}^{j-1}(x_{j-1}-x_\nu)\prod_{\nu=j}^{n}(x_\nu-x_{j-1})$. There is also a recurrence formula by means of which V_{n+1}^{-1} can be calculated if V_n^{-1} is known. *H. Schwerdtfeger.*

Kuntzmann, Jean. Etude de représentations approchées dans le cas de deux variables. Chiffres **1** (1958), 35–39.

L'auteur appelle „silhouette" un ensemble de points

à coordonnées entières i, j (positives ou nulles), qui, avec le point (i, j) contient tous les points (i', j') tels que $i' \leq i$, $j' \leq j$. Cette notion est d'abord appliquée à des polynômes à deux variables $\sum x^i y^j$, puis on démontre que, si L désigne une fonctionnelle linéaire, l'ensemble des couples (i, j) tels que: $L(x^i y^j)=0$ forme une silhouette. L'auteur donne enfin sans démonstration les silhouettes correspondant aux représentations approchées du laplacien et d'une dérivée partielle d'une fonction de deux variables, représentations dont le reste est une fonctionnelle linéaire.

G. Brillouet (Nantes).

Samssonow, Victor. Suite récurrente à convergence rapide tendant alternativement par excès ou par défaut vers $e^{2/n}$. C. R. Acad. Sci. Paris **246** (1958), 1152–1153.

The sequence $U_{-1}=a_{-1}/b_{-1}=1/-1$, $U_0=a_0/b_0=1/1$, $U_p=a_p/b_p=[(2p-1)na_{p-1}+a_{p-2}]/[(2p-1)nb_{p-1}+b_{p-2}]$ $(p=1, 2, \cdots)$, containing the fixed parameter n, converges to the limit $U_\infty(n)=e^{2/n}$. The rapidity of convergence is discussed.

E. Frank (Chicago, Ill.).

Riabouchinsky, Dimitri. Observations au sujet de la Note de M. Victor Samssonow. C. R. Acad. Sci. Paris **246** (1958), 1153.

From the recurrence formula of Samssonow [see the paper reviewed above] it follows that $a_p(n)=\pm b_p(-n)$, $b_p(n)=\pm a_p(-n)$, where the $+$ or $-$ sign corresponds to even or odd values of p, $U_p(n)U_p(-n)=1$, and $U_p(-n) \to e^{-2/n}$. Consequently, his formula is easily applicable to positive and negative real values of n. *E. Frank.*

Stiefel, Eduard L. Kernel polynomials in linear algebra and their numerical applications. Nat. Bur. Standards Appl. Math. Ser. no. **49** (1958), 1–22.

The present paper gives the theoretical background of those aspects of the theory of orthogonal polynomials and the associated kernel polynomials which are basic in their applications to numerical analysis. In particular, their applications to the problem of solving linear equations and to the determination of eigenvalues and eigenvectors are discussed both from a theoretical point of view and from the point of view of their use in high speed computation. The quotient difference algorithm is developed and its applications are explained. The material given in this paper is basic in numerical analysis and is excellent subject matter for courses on matrix theory in connection with numerical analysis. *M. R. Hestenes*

Hestenes, Magnus R. Inversion of matrices by biorthogonalization and related results. J. Soc. Indust. Appl. Math. **6** (1958), 51–90.

The paper takes up a variety of topics, starting with a method of inversion and proceeding to problems calling for an inversion at some stage. If U and V are non-singular matrices, the method presented for inverting U is equivalent to the application of Jordan eliminations to VU, and the name of the method arises from the observation that the rows of U^{-1} and the columns of U form a pair of biorthogonal systems. Hence the rows of V are to be replaced by linear combinations which will exhibit the necessary biorthogonality. In case V is nearly equal to U^{-1}, hence VU nearly equal to I, the rounding errors may be presumed minimal. The case of U singular, possibly rectangular, is given some attention and has rather obvious applications.

A quadratically convergent iteration to a proper value and the associated vectors is described which involves bordering $A-\lambda_k I$ by the current approximations to these vectors, and inverting the resulting matrix.

Most of the latter half of the paper is concerned, directly or indirectly, with the proper values and vectors of UU^* and U^*U. It is legitimate to assume U to be square, although this is not done, since one can border by null rows or columns if necessary. Let $X^*UU^*X=Y^*U^*UY=\Lambda^2$, where Λ^2 is diagonal, X and Y unitary. There is no restriction in supposing the diagonal elements of Λ non-negative, and non-null in the first r places, where r is the rank of U. Let $\Lambda^{(-1)}$ represent the matrix in which the non-null elements of Λ are replaced by their reciprocals. It is known that $U=X\Lambda Y^*$; then $U^{(-1)}=Y\Lambda^{(-1)}X^*$ possesses many of the properties of an inverse, and is, in fact, E. H. Moore's "general reciprocal." The author calls the diagonal elements of Λ the principal values of U, and the columns of X and of Y principal vectors (to this reviewer "singular" seems to conform better than "principal" with previous usage). A method of iterated rotations is described [see also Kogbetliantz, Quart. Appl. Math. **13** (1955), 123–132; MR **19**, 581], for obtaining the principal values and vectors, and geometric properties of the generalized inverse are discussed.

A. Householder (Oak Ridge, Tenn.).

★ **Mitra, S. K. On an orthogonalisation method of evaluating the reciprocal and the determinant of a matrix and its Gaussian transform.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 261–268. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

Applies orthonormalization to the columns of A; then observes that square roots can be avoided by merely orthogonalizing. *A. Householder.*

Heller, J. Ordering properties of linear successive iteration schemes applied to multi-diagonal type linear systems. J. Soc. Indust. Appl. Math. **5** (1957), 238–243.

Let the matrix A be partitioned in the form (A_{ij}), where each A_{ii} is square. For $i<j$, let $\tau_{ij}(1-\tau_{ij})=0$ and form the determinant of the matrix obtained from A when each A_{ii} is multiplied by λ, each A_{ij} by $\lambda^{\tau_{ij}}$, and each A_{ij} by $\lambda^{1-\tau_{ij}}$. Any selection of the τ_{ij} corresponds to a particular ordering of the equations for a single-step iteration scheme, and the λ of largest modulus for which the determinant vanishes governs the rate of convergence. An ordering is "equivalent" in case the determinant has the same zeros as with $\tau_{ij}=0$. The main theorem gives a necessary and sufficient condition for equivalence, and discursive comments are made about certain special cases.

A. S. Householder (Oak Ridge, Tenn.).

Schmidtmayer, J. Über die Auflösung des Systems linearer, algebraischer Gleichungen mit komplexen Koeffizienten. Z. Angew. Math. Mech. **38** (1958), 74–77.

Let $pz=q$ be a set of m equations with n unknowns; the matrix p is $m \times n$ with complex entries. By setting $p=a+ib$, $z=x+iy$, $q=u+iv$, the system reduces to the pair of real matric equations $ax-by=u$, $bx+ay=v$. The author calls this the usual procedure and points out that even when p is non-singular, both a and b may be singular;

also difficulties arise if some of the given equations are dependent.

The author suggests that each entry p_{jk} in p be given its usual representation as a 2×2 matrix with real entries. The given equations are then replaced by the equivalent set $PZ=Q$, where P is $2m\times2n$. Elementary operations on the matrix P will then reduce the set of equations to a simpler form; in the case that P is square and non-singular, p^{-1} can be found by the standard process of reducing P to the identity and concomitantly performing the same operations on the identity.

Two examples are given in which p is a 3×3 matrix with entries which are small integers; no evidence is given to support the author's statement that the method is suitable for machine calculation. *R. G. Stanton.*

Falk, Sigurd. Das Ersatzwertverfahren als Hilfsmittel bei der iterativen Bestimmung von Matrizen-Eigenwerten. Abh. Braunschweig. Wiss. Ges. **8** (1956), 99–110.

Das Iterationsverfahren bei algebraischen Eigenwertaufgaben konvergiert, wenn überhaupt, gegen den (oder einen) Eigenwert mit maximalem Betrage. Es wird ein Verfahren angegeben, mit dessen Hilfe auch andere Eigenwerte iterativ bestimmt werden können, wenn der Eigenwert mit maximalem Betrage nebst zugehörigen Eigenvektoren bereits bekannt ist.

Zusammenfassung des Autors.

Freire, Rémy. A matricial method for the solution of certain systems of linear equations. Soc. Parana. Mat. Anuário **3** (1956), 54–59. (Portuguese)

Remarks on the possibility of solution by breaking the coefficient matrix into blocks.

Gouarné, René. Remarques sur le calcul automatique des déterminants et polynomes caractéristiques par la méthode des cycles. C. R. Acad. Sci. Paris **245** (1957), 1998–2000.

On donne un organigramme du calcul de $|A|$ pour une matrice $n\times nA$ qui obtient les termes sous forme naturelle [un terme est sous cette forme quand ses éléments facteurs sont dans l'ordre des indices de ligne croissants, par opposition à la forme cyclique, expression du terme comme produit de termes cycliques (1)]. On introduit cet organigramme dans celui (2) de $|A-\lambda I|$ et l'on compare les deux procédés. [Résumé de l'auteur] *R. W. Hamming.*

Sabroff, Richard R.; and Higgins, T. J. A critical study of Kron's method of "tearing". III. Matrix Tensor Quart. **8** (1957), 43–51.

Givens, Wallace. Computation of plane unitary rotations transforming a general matrix to triangular form. J. Soc. Indust. Appl. Math. **6** (1958), 26–50.

This paper discusses a number of preparatory operations on a complex matrix A after which a numerical computation of its eigen values and eigen vectors by means of computing machines becomes reasonably simple. — An $m\times n$-matrix A is said to be p-subtriangular if $a_{ij}=0$ for $i>j+p$, and p-supertriangular if $a_{ij}=0$ for $j>i+p$, where p is an integer. A p-subtriangular matrix A is complete if $a_{ij}\neq0$ for $i=j+p$. It is shown that for any row vector x, a unitary matrix U, acting effectively in a single coordinate plane x_i, x_j only, can be found such that the one coordinate of xU in this plane is zero. The "plane component" of U may be written as a "unitary rotation", viz. $\begin{pmatrix} c & -s \\ \bar{s} & c \end{pmatrix}$, and the corresponding $n\times n$-matrix may be denoted by R_{ij}. Accordingly, a succession of unitary rotations $R_{1,k}$ $(k=2,\cdots,n)$ can be constructed such that $xR_{12}R_{13}\cdots R_{1n}=(|x|,0,\cdots,0)$. Similarly [as the author has shown; see the review below] $\frac{1}{2}(n-1)(n-2)$ such "rotations" can be found so that if R is their product, the matrix A is reduced into a one-subtriangular matrix $S=R^*AR$. If z of the subdiagonal elements of this matrix are zero, then S has at most $z+1$ linearly independent characteristic vectors associated with any characteristic root; in particular, a complete one-subtriangular matrix has a uniqueeigen vector for each eigenvalue. The theorem is generalized to a p-subtriangular matrix S $(p>0)$. If $S=(S_{ij})$ is a one-subtriangular "block-matrix" which has, along its diagonal, a system of one-subtriangular square matrices $S_{11}=S_1$, $S_{22}=S_2$, $\cdots$, then the roots of S are those of S_1, S_2, $\cdots$, and are independent of the blocks S_{ij} $(i<j)$.

Another type of simplification that can be reached for a one-subtriangular S is the following: Unitary rotations $R_{n,k}$ can be found such that the last $n-k$ elements in the last column of $S_\lambda^{(k)}$ are zero, where $S_\lambda^{(n)}=S-\lambda I$, $S_\lambda^{(k)}=S_\lambda^{(k+1)}R_{n,k}$ $(k=n-1,\ n-2,\ \cdots,\ 2,\ 1)$, and each $S_\lambda^{(k)}$ is one-subtriangular. In particular, $S_\lambda^{(1)}$ is a matrix whose last column consists of zeros, except possibly the first element $g=g(\lambda)$; if S is complete, then $g(\lambda)=0$ if and only if λ is a characteristic root of S. If $g(\lambda)\neq0$, but λ is a characteristic value of S, then the reduction of the last column of $S-\lambda I$ into zeros can be completed by a right-hand multiplication by unitary rotation matrices $R_{q,j}$, $R_{n,q}$, if a subdiagonal element $r_q=s_{q+1,q}$ of S vanishes. Based upon this result, it is shown that by a unitary transformation, a product of rotations in coordinate planes, the matrix S can be reduced to $T=R^*SR=T_\lambda+\lambda I$, where all elements of the last column vanish, except the last one which equals λ. Finally, there is a discussion of the general nature of a matrix product of the form $R_{n,n-1}R_{n,n-2}\cdots R_{n,1}$. To satisfy the prospective computer, an estimate is given of the number of numerical multiplications which are necessary to carry out every single step in the described matrix operations.

H. Schwerdtfeger (Montreal, P.Q.).

Givens, Wallace. The characteristic value-vector problem. J. Assoc. Comput. Mach. **4** (1957), 298–307.

The author's method for obtaining the characteristic values of a symmetric matrix is now well known and widely used [Oak Ridge Nat. Lab. Rep. **ORNL-1574** (1954); MR **16**, 177]. It is based upon the fact that by a sequence of $\frac{1}{2}n(n-1)$ plane rotations, a symmetric matrix can be transformed to tridiagonal form. The present paper proposes the application of a similar sequence of rotations to transform an arbitrary real matrix to a form in which all elements below the subdiagonal are null. A variety of suggestions are made for completing the problem, one being to compute the characteristic polynomial by a recursion; another to transform the matrix to a form analogous to the companion matrix except that elements along the subdiagonal are not restricted. Operational counts are given. *A. S. Householder.*

Urabe, Minoru. Convergence of numerical iteration in solution of equations. J. Sci. Hiroshima Univ. Ser. A. **19** (1956), 479–489.

R is a linear normed space and F a complete subset of R. T is a functional defined in F such that $T(F)\subset R$. It is assumed that T is Lipschitz bounded, viz. there exists a

positive constant K such that $\|Tf_1-Tf_2\|\leqq K\|f_1-f_2\|$ for any $f_1, f_2 \in F$.

In F, the equation (I) $x=Tx$ is considered, and it is assumed that (I) $K<1$; (II) for a selected $x_0 \in F$, $x_1=Tx_0$ belongs to F; (III) the sphere

$$S\left\{h:\|h-x_1\|\leqq\frac{K}{1-K}\|x_1-x_0\|\right\}$$

is contained in F. L. Collatz has shown that the iteration $x_{n+1}=Tx_n$ ($n=0, 1, 2, \cdots$) can be continued indefinitely and the sequence $\{x_n\}$ converges to a certain limit $\bar{x}$, which give a unique solution of (I) in F [Z. Angew. Math. Phys. **4** (1953), 327–357; MR **15**, 559]. But when x is a numerical quantity, there arises an error in computation of Tx, consequently the obtained sequence is not $\{x_n\}$, but the numerical sequence $\{x_n^*\}$ determined by $x_{n+1}^*=T^*x_n^*$, where T^* is an approximate functional of T.

The author shows that the numerical sequence $\{x_n^*\}$ does not necessarily converge contrary to convergence of the true sequence $\{x_n\}$.

In this paper the following problems are discussed. 1) At what step should the iteration process (1) be stopped? 2) When the iteration process is stopped at the favorable step in this sense, with how much error is the obtained approximate solution attended?

It is shown, under general conditions of boundedness, that the relation $x_{n+m}^*=x_n^*$ surely occurs for certain $n\geqq N$ and $m>0$. $x_{n+m+i}^*=x_{n+i}^*$ ($i=0, 1, 2, \cdots, m-1$), so the sequence $\{x_n^*\}$ oscillates, taking m values. This is called the state of numerical convergence of the sequence $\{x_n^*\}$. The case where the algebraic or transcendental equations are solved by the method of iteration falls under this case. The special case $x=\varphi(x)$, where R is a real line, $\varphi(x)$ continuous, with its derivative of the first order in the closed interval F leads to the conclusion: When the computation is carried on so minutely that the bound of errors may be sufficiently small and $\varphi(x)$ is computed so accurately that the error may not exceed that bound, after finite times of repetition of iteration the numerical sequence $\{x_n^*\}$ necessarily attains the state of numerical convergence where x_n^* takes at most two values. *S. C. van Veen* (Delft).

Rodeja, E. G. Study of a group of methods of numerical solution of algebraic equations. Mem. Mat. Inst. "Jorge Juan" no. **17** (1956), i+144 pp. (Spanish)

This study of a group of methods of numerical solution of algebraic equations contains a proof and detailed discussion and generalization of the following fundamental theorem: Let $f(x)$ be a polynomial of degree n with real or complex coefficients. Multiply $f(x)$ by a polynomial $\phi_p(x)$ of degree p. Then, dependent on the choice of $\phi_p(x)$, the product $F_p(x)=f(x)\phi_p(x)$ has all its coefficients equal to zero except $n+1$ special coefficients. The results obtained are related to those of Sebastião e Silva, Wronski, Gräffe, and many others. The book also contains a historical development of methods of solution of algebraic equations, and a bibliography of 112 references on the subject. *E. Frank* (Chicago, Ill.).

Ward, James A. The down-hill method of solving $f(z)=0$. J. Assoc. Comput. Mach. **4** (1957), 148–150.

This paper describes an approximation to a root of an analytic equation $f(z)=0$, based upon the theorem "If $f(x+iy)=R+iJ$ (R, J real), then the real non-negative function $W=|R|+|J|$ has no minimum value different from zero" and on a discussion of the surface $u=W(x, y)$ in the cartesian x, y, u coordinate system. There are remarks on the programming of the "down hill" method on the CRC 102A. *H. Schwerdtfeger.*

Heigl, F. Über die Abschätzung der Wurzeln algebraischer Gleichungen. Monatsh. Math. **62** (1958), 16–55.

A number of upper and lower bounds are derived for the absolute values of the roots of an algebraic equation. The author's bounds improve bounds previously obtained by S. Kakeya [Tôhoku Math. J. **2** (1912), 140–142], M. Fujiwara [ibid. **10** (1916), 167–171], J. L. Walsh [Ann. of Math. (2) **25** (1924), 285–286], T. Anghelutza [Boll. Un. Mat. Ital. **12** (1933), 284–289], T. H. Chang [Tôhoku Math. J. **43** (1937), 79–83], particularly if the polynomial in question has gaps and the minimum gap length is larger than 1.

The following is a typical result. Let the equation be $\sum_{\mu=0}^{\varkappa} a_{s_\mu} z^{n-s_\mu}=0$, $0=s_0<s_1<\cdots<s_\varkappa=n$, with non vanishing real or complex coefficients a_{s_μ} and $a_0=1$. Let $s=\mathrm{Min}(s_1, s_2-s_1, \cdots, s_\varkappa-s_{\varkappa-1})$, $n=is+k$, $0\leqq k<s$, $A_{s_\mu}=|a_{s_\mu}|$. Then the absolute value of each root is at most equal to

$$2^{1/s}\,\mathrm{Max}\{A_{s_\lambda}^{1/s_\lambda}, [2^{-1-(k/s)}A_{s_\varkappa}]^{1/s_\varkappa}\}\ (\lambda=1, 2, \cdots, \varkappa-1)$$

and at least equal to

$$2^{-1/s}\,\mathrm{Min}\{[2^{1+(k/s)}A_{s_\lambda}]^{1/s_\lambda}, (A_{s_\lambda}/A_{s_{\lambda-1}})^{1/(s_\lambda-s_{\lambda-1})}\}$$

$$(\lambda=2, 3, \cdots, \varkappa).$$

For equations with real coefficients the author also generalizes a well known result of A. Hurwitz [ibid. **4** (1913), 89–93] as well as a generalization of it given by J. Heinhold [Monatsh. Math. **59** (1955), 203–216; MR **17**, 138]. Attached to the paper is an extensive bibliography with 132 references on related work.

Walter Gautschi (Washington, D.C.).

Simonsen, W. On formulae for numerical differentiation. Skand. Aktuarietidskr. **40** (1957), 60–70.

If a function $f(x)$ is known at $n+1$ arguments $x_0<x_1<\cdots<x_n$ its mth derivative ($m\leqq n$) may be evaluated by differentiating Lagrange's interpolation formula m times. J. F. Steffensen [Interpolation, Williams-Wilkins, Baltimore, 1927] gives a corresponding error term of the form $Af^{(n+1)}(\xi)$ under the assumption that x is not in the interior of the interval (x_0, x_n). Here the author shows that for x lying in certain subintervals of (x_0, x_n), which he specifies, error terms of the form $Af^{(n+1)}(x_0)+Bf^{(n+2)}(\xi)$ or $Af^{(n+1)}(x_n)+Cf^{(n+2)}(\eta)$ are valid. Conditions are discussed as to where the first and where the second form applies. *Walter Gautschi* (Washington, D.C.).

Grunsky, Helmut. Eine Methode zur Lösung von Anfangswertproblemen bei gewöhnlichen und partiellen linearen Differentialgleichungen zweiter Ordnung. Z. Angew. Math. Mech. **34** (1954), 291–292.

Given $y''+a(x)y'+b(x)y=0$, $y(x_0)=y_0$, $y'(x_0)=y_0'$, the author writes $y(x)=y_0+y_0'(x-x_0)+\int_{x_0}^{x}(x-\xi)y''(\xi)d\xi$, replaces y'' in the integrand by $-(ay'+by)$ and brings in y'' again by integrations by parts. The resulting expression can be made free of terms containing $y'(x)$ by a suitable choice of the constants in the quadratures involved. The method referred to in the title consists of an iteration of this process. An analogous method is described for second order partial differential equations of hyperbolic type. *Walter Gautschi* (Washington, D.C.).

Schäfke, Friedrich Wilhelm. Zur Integrationsmethode von Herrn Grunsky. Arch. Math. 8 (1957), 125–128.

Der Verf. überträgt ein Verfahren von H. Grunsky [oben referiert] zur Behandlung von gewöhnlichen und partiellen Differentialgleichungen auf Gleichungen der Form (*) $y'(\tau)=A(\tau)y(\tau)$ in einem Banachschen Raum und beweist einen Konvergenzsatz. $A(\tau)$ bedeutet eine im Intervall $[\alpha, \beta]$ stetig von τ abhängende, stetige lineare Abbildung des Banachschen Raumes in sich. Mit willkürlich gewählten Operatoren $B_\nu(\tau_1)$ ($\nu=1, 2, \cdots$) und $B_0(\tau)=E$ (=Einheitsoperator) sei $B_\nu(\tau)=B_\nu(\tau_1)+\int_{\tau_1}^{\tau} B_{\nu-1}(\xi)A(\xi)d\xi$. Gilt $\|B_\nu(\tau_1)\|\to 0$ ($\nu\to\infty$), so hat man für jede Lösung von (*), $0=\sum_{\nu=0}^{\infty}\{(-1)^\nu B_\nu(\tau)y(\tau)|_{\tau_1}^{\tau_2}\}$ ($\tau_1, \tau_2\in[\alpha, \beta]$). Ist ausserdem eine der beiden Reihen in (*) konvergent, so auch die andere, und man erhält (*) $\sum_{\nu=0}^{\infty}(-1)^\nu B_\nu(\tau_1)y(\tau_1)=\sum_{\nu=0}^{\infty}(-1)^\nu B_\nu(\tau_2)y(\tau_2)$.

J. Schröder (Hamburg).

Sheldon, J. W.; Zondek, B.; and Friedman, M. On the time-step to be used for the computation of orbits by numerical integration. Math. Tables Aids Comput. 11 (1957), 181–189.

The main aim of this investigation has been to throw some light on the optimum time step to be adopted in extended numerical integrations of mass-particle orbits in dynamical astronomy. To this end, Eckert's modification of Cowell's method is adopted without discussion, and the cumulative effects of the truncation errors entailed in its use are compared with round-off at the end of a desired time interval in order to judge the optimum length of single steps which equalizes them. From an analysis of some particular cases, the authors conclude that, if one employs a going-ahead formula involving tabular differences of orders up to the 10th, some 60 steps per revoluti n would appear to furnish the optimum results. *Z. Kopal.*

Schröder, Johann. Über das Differenzenverfahren bei nichtlinearen Randwertaufgaben. I. Z. Angew. Math. Mech. 36 (1956), 319–331. (English, French and Russian summaries)

The author describes the simplest iteration method for solving the standard finite difference equations that approximate the boundary value problems of second order ordinary differential equations. He gives special attention to equations: A: $\phi''+f(x, \phi)=0$; A': $\phi''+f(x, \phi, \phi')=0$; B: $\phi''+K\phi'+c\phi+f(x, \phi)=0$; B': $\phi''+K\phi'+c\phi+f(x, \phi, \phi')=0$; for $0\leqq x\leqq l$, with different appropriate boundary conditions. The function f, the constants K, c and l, as well as the parameters in the boundary conditions, are restricted so as to insure that the standard iteration scheme converges. The author estimates the error in the iterated solutions. Numerical examples are given. *E. Isaacson* (New York, N.Y.).

Proskuryakov, A. P. On the construction of periodic solutions of autonomous systems with one degree of freedom. Prikl. Mat. Meh. 21 (1957), 585–590. (Russian)

Equation $\ddot{x}+k^2x=\mu f(x, \dot{x}, \mu)$, f analytic, is solved by means of the change of variables $t=k^{-1}h\tau$, $h=\sum h_i\mu^i$ ($h_0=1$), $x(\tau)=a(\tau)\cos\tau+b(\tau)\sin\tau$, $x'(\tau)=-a(\tau)\sin\tau+b(\tau)\cos\tau$, $a(\tau)$, $b(\tau)$ being power series in μ, the coefficients of which are periodic functions of τ of period 2π. In the differential system for a, b, equal powers of μ are identified; the periodicity conditions for the coefficients of the series of a, b then yield equations which may be solved successively for these coefficients as well as for the numerical coefficients h_i. The equations are written explicitly up to terms in μ^2. The method is applied to van der Pol's equation. *J. L. Massera* (Montevideo).

★ **Diaz, J. B. Upper and lower bounds for eigenvalues. Calculus of variations and its applications.** Proceedings of Symposia in Applied Mathematics, Vol. VIII, pp. 53–78. McGraw-Hill Book Co., Inc., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1958. 153 pp. $7.50.

This paper is an excellent summary (as of April, 1956) of the art of eigenvalue approximation with arbitrary known accuracy. After a discussion of Courant's maximum-minimum definition of eigenvalues and the resulting monotonicity with respect to admissible functions, the Rayleigh-Ritz method for upper bounds, and the method of A. Weinstein for lower bounds for the eigenvalues of partial differential operators are treated in detail. The last section refers to several more recent results. In particular, a new proof is given for the strict inequality $\omega_n^2<\lambda_n^2$, where ω_n is the nth eigenvalue of any clamped membrane and λ_n^2 is that of the clamped plate of the same shape. (The function u in formula (116) should, of course, be v.) *H. F. Weinberger* (College Park, Md.).

Koval', P. I. Sur la stabilité des solutions approximatives des équations différentielles paraboliques et hyperboliques. Ukrain. Mat. Ž. 9 (1957), 271–280. (Russian. French summary)

The method of differences applied to the parabolic equation (*) $u_t=a(x, t)u_{xx}+b(x, t)u_x+c(x, t)u$, $u=u(x, t)$, $0\leqq x\leqq H<\infty$, $t\geqq 0$, leads to a system which can be written in the form (**) $u_{s+1}=(E+\mu A_s)u_s+b_s$. The author proves that if the parameter $\mu=l/h^2$, where $h=\Delta x$, $l=\Delta t$ are the mesh sides, is chosen to be less than $2/(-\varphi)$, and if $\Phi<0$, then the solutions of (**) are asymptotically stable; if, however, $\Phi>0$, the solutions are unstable for every value of μ. Here Φ, φ are, respectively, the largest and smallest of the eigenvalues of the matrix $A=\lim_{s\to\infty}A_s$. If, instead of the first difference, a Lagrange polynomial is used to approximate the derivative u_t, the resulting system corresponding to (**) yields only unstable solutions to the differential equation.

For the hyperbolic equation obtained by replacing the left member of (*) by u_{tt}, the parameter $\mu=l^2/h^2$ can be chosen in such a way that solutions of the associated system, mutatis mutandis, are stable if $\Phi<0$. Use of the Lagrange approximation for u_{tt} results in stability for only one particular set of mesh parameters.

R. N. Goss (San Diego, Calif.).

Dumitrescu, D.; Ionescu, V.; und Toth, R. Die Anwendung des Differenzverfahrens zum Studium der Bewegung schwerer Flüssigkeiten mit freier Oberfläche. Rev. Méc. Appl. 1 (1956), no. 2, 43–81. (1 plate)

Mit dem gewöhnlichen Differenzenverfahren werden verschiedene Fälle der (stationären, wirbelfreien) Bewegung schwerer (inkompressibler) Flüssigkeiten untersucht. — Zunächst wird die Strömungsgeschwindigkeit u einer schweren und zähen Flüssigkeit in geneigten Kanälen konstanten Querschnittes bei Vernachlässigung der Reibung zwischen Luft und Flüssigkeit berechnet, und zwar für verschiedene dreieckige und aus halben Sechsecken bestehende Querschnitte. Dieses Problem führt auf die erste Randwertaufgabe für die Differentialgleichung $\partial^2u/\partial y^2+\partial^2u/\partial z^2=K$ (= const.). Die verwendeten Gitter bestehen aus Quadraten, Rechtecken oder Sechsecken.

— Weiter behandeln die Verf. die Bewegung schwerer idealer Flüssigkeiten mit freier Oberfläche, und zwar in einer senkrechten Ebene (1) die Strömung über einen ebenen unvollkommenen Überfall und (2) die Strömung über eine Stauklappe in Form eines Kreisbogens und als achsialsymmetrische Strömung (3) die aufsteigende Bewegung einer Luftblase in einem senkrechten Rohr.

Mathematisch handelt es sich hier nicht um Randwertaufgaben im üblichen Sinne. Die Differentialgleichung für die Stromfunktion ψ lautet $\partial^2\psi/\partial y^2+\partial^2\psi/\partial z^2=0$ bei (1) und (2), $\partial^2\psi/\partial z^2+\partial^2\psi/\partial r^2-(1/r)\partial\psi/\partial r=0$ bei (3). Auf dem freien Rand als Stromlinie ist ψ vorgeschrieben. Jedoch ist dieser Rand selbst nicht bekannt; er muss so bestimmt werden, dass auf ihm die Bernoullische Beziehung gilt. Dies führt auf quadratische Gleichungen in den Näherungswerten für ψ. Praktisch wird der freie Rand zunächst vorgegeben und das zugehörige Dirichletsche Problem mit dem Differenzenverfahren gelöst. Dann wird der freie Rand so lange geändert, bis die entsprechenden Näherungslösungen die geforderten Beziehungen erfüllen. — Die Lösung der linearen Differenzengleichungen geschieht mit der Relaxationsmethode von groben zu feineren Gittern übergehend. *J. Schröder* (Hamburg).

Conte, S. D. A stable implicit finite difference approximation to a fourth order parabolic equation. J. Assoc. Comput. Mach. **4** (1957), 18–23.

The differential equation $\partial^4U/\partial x^4+\partial^2U/\partial t^2=0$ is approximated by the difference equation

$$u_{j+2,k}-(u_{j+1,k+1}+2u_{j+1,k}+u_{j+1,k-1})+2(u_{j,k+1}+u_{j,k}+u_{j,k-1})-(u_{j-1,k+1}+2u_{j-1,k}+u_{j-1,k-1})+u_{j-2,k}+r^{-2}(u_{j,k+1}-2u_{jk}+u_{j,k-1})=0,$$

where $u_{jk}=u(j\Delta x, k\Delta t)$, and $r=\Delta t/\Delta x^2$. The solution of an initial-boundary value problem for this implicit difference equation in a finite x-interval involves, at every line $t=j\Delta t$, the solution of a system of linear equations whose matrix is tri-diagonal and therefore simpler than previously proposed implicit schemes [cf. Royster and Conte, Proc. Amer. Math. Soc. **7** (1956), 742–749; MR **17**, 1214]. It is shown that this difference scheme is unconditionally stable in the sense that all solutions with simple harmonic initial values decrease with increasing t, no matter what value of r is chosen. If U satisfies the boundary and initial conditions $U(0,t)=U(1,t)=U_{xx}(0,t)=U_{xx}(1,t)=0$, $U(x,0)=f(x)$, $U_t(x,0)=0$, and if analogous conditions are imposed on u, both U and u can be explicitly represented by Fourier sine series. Assuming that $f(x)\in C^2$, that f''', $f^{(4)}$ are piecewise continuous, and that $f(0)=f(1)$, the author proves, by means of these representations, that u tends uniformly to U, for constant r, as Δx^2 tends to zero.

W. Wasow (Madison, Wis.).

Atta, Susie E. Effect of propagated error on inverse of Hilbert matrix. J. Assoc. Comput. Mach. **4** (1957), 36–40.

Elements of the sixth-order segment of the Hilbert matrix are rounded in various ways and the resulting matrix inverted. This computed inverse is compared with the known inverse of the segment. Also the value is computed for the right member of the inequality

$$\|A^{-1}-B\|\leqq\|B\|\,\|C\|/(1-\|C\|),$$

where B is the computed inverse and $C=I-AB$, and disagreement arises, presumably due to the fact that on the left in this inequality one does not know A^{-1}, which would be the inverse of the rounded segment, but only the inverse of the true segment.

More interesting is the fact that when the segment is taken to six significant decimal digits only, then only one element of the computed inverse agrees even in sign with the true inverse. *A. S. Householder.*

See also: Linear Algebra: Herz. **Computing Machines:** Gouarné. **Statistics:** Akaike. **Mechanics of Particles and Systems:** Duffin and Schild. **Elasticity, Plasticity:** Kato, Fujita, Nakata and Newman. **Fluid Mechanics, Acoustics:** Guilinger and Saibel; Belotserkovsky; Chushkin. **Economics, Management Science:** Peck and Hazelwood.

Computing Machines

Shannon, Claude E. Von Neumann's contributions to automata theory. Bull. Amer. Math. Soc. **64** (1958), 123–129.

The paper is one of a series of memorials to the memory of John von Neumann. The author gives an account of von Neumann's work in the Theory of Automata. He mentions both his published and unpublished work in this area. *H. H. Goldstine* (Princeton, N.J.).

★ **Chapin, Ned. An introduction to automatic computers.** D. Van Nostrand Company, Inc., Princeton-Toronto-London-New York, 1957. viii+525 pp. (1 plate) $8.75.

This is a descriptive introduction to electronic computers, including their method of working, installation and operational problems, and use in business data processing. It is directed to persons with business training. There is no mathematics; however, the essentials of programming and coding are presented, in the discussion on flow diagrams and in the coding of three task problems on hypothetical computer. There are a glossary and three appendices; the last of these contains a detailed listing and description of commercially available computers.

C. C. Gotlieb (Toronto, Ont.).

Reitwiesner, George W. Performing binary multiplication with the fewest possible additions and subtractions. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md. vol. 4 (1957), no. 3, pp. 11–14. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.)

The representation of a non-negative integer in conventional number-representation systems is unique to within redundant terminal zero-symbols. If the digits are permitted to range over negative as well as positive values, uniqueness is forfeited. On the other hand, when performing multiplication by successive additions, the possibility of negative digits permits the replacement of some of the additions by subtractions. For a given number, it is of interest to find a representation such that the sum of the absolute values of the digits is a minimum. Such a representation for a multiplier will be advantageous, since the addition-subtraction time will be least. The author here summarizes a "rigorous and more lengthy argument intended for publication in a suitable technical journal." It is stated that in the binary system, (1) on the

average, this procedure reduces the number of addiitons-subtractions from one-half the total number of digits to approximately one-third, and (2) the representation with minimum sum of absolute values of digits is unique.

Recursive formulas are given for computing the coefficients of the latter representation from the digits of the representation in "2's complement, 1's complement, and magnitude with appended sign." Possibly because of space limitations no proofs of the above-stated results are given.

G. W. Patterson (Philadelphia, Pa.).

Valach, Miroslav. Origin of the code and number system of remainder classes. Stroje na Zpracování Informací 3 (1955), 211–245 (1956). (Czech. Russian and English summaries)

For some applications, it is of great advantage to have digital computer circuits which operate in a single cycle, rather than serially, when evaluating arithmetical expressions. For multiplication, this requires a great increase in complexity of the hardware. However, substantial savings can be achieved by the use of a special code, called the H-code, for the representation of the digits in the decadic number system. Such a code is described. It enables single-cycle addition, multiplication and even the evaluation of entire algebraic expressions which involve only these operations. The code is related to the theory of remainder classes.

The paper describes a single-cycle binary adder, and a multiplier, as well as the properties which the H-code must possess, the relation between the H-code and the represented number, the number system of remainder classes and the evaluation of algebraic expressions in this system.

V. Vand (University Park, Pa.).

Balasiński, W.; and Mrówka, S. On algorithms of arithmetical operations. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 803–804, LXVIII. (Russian summary)

The expansion of any number ξ which is a result of arithmetical operations with numbers α, β, etc., in terms of powers of $-g$, where g is a positive integer, $\xi=\sum_{i=-N}^{n} a_i(-g)^{-i}$ (N positive or negative integer) can be computed by making use of the inequalities: if N is odd, $(a_N\neq 0)$ $a_N/(-g)^N+1/[(-g)^N(g+1)]\leq\xi\leq a_N/(-g)^N+1/[(-g)^{N-1}(g+1)$ and if N is even $a_N/(-g)^N+1/[(-g)^{N-1}(g+1)]\leq\xi\leq a_N/(-g)^N+1/[(-g)^N(g+1)]$. By substituting the original number α, β, etc., into these inequalities, one can find a_N. By repeating the procedure with $\xi_1=\xi-a_N/(-g)^N$, one then finds the next coefficient, etc.. As an application, a division algorithm is worked out for $g=2$ requiring only additions, subtractions and shifting by one position. {Reviewers note: If β and α are both positive or both negative, then the inequalities $r^{(i+1)}\leq 4r_i$ given in the article should be turned around.}

U. Hochstrasser (Lawrence, Kans.).

Friedberg, R. M. A learning machine: I. IBM J. Res. Develop. 2 (1958), 2–13.

The problem treated here is that of getting a machine to learn to do things without being told precisely how. Assuming a test for satisfactory performance, this can be done, in principle, by generating and testing a variety of computer programs. But this raises a basic heuristic question: the universe of possible computer programs, even with modest bounds on size, is simply too vast to fully explore, even with the most powerful available computers. Friedberg's attempt to reduce the amount of search involves the equivalent of dividing the space of programs into large classes, such that the success or failure of a member can be interpreted as a reflection on the others of the same class. The success of this approach depends critically on the choice of the classification or measure of 'similarity' used.

The trial programs consist of strings of 64 pseudo-instructions which are interpreted by a program within the IBM 704 computer. A second program is supplied with a success-test in which is embodied the bit-manipulation task to be learned. A third program embodies a kind of reinforcement learning process which alters the pseudo-instructions of the trial programs in accord with certain records concerning past experience.

The reinforcement process is such that two programs are treated similarly to the extent that they contain the same instructions in the same registers or locations. As the author points out, programmers know that this is a rather superficial notion of similarity for programs, and one would tend to doubt its adequacy. These doubts will remain after examination of the first experimental results. The author suggests a number of directions in which the system might be improved. The reviewer hopes that the rather inconclusive experimental result will not divert attention from the author's excellent general discussion of this approach.

M. L. Minsky.

Jeenel, J. Programs as a tool for research in systems organization. IBM J. Res. Develop. 2 (1958), 105–122.

The author discusses the effect of type of storage and rearrangement of data in storage on the time efficiency of a computer program.

S. Gorn (Philadelphia, Pa.).

Gouarné, René. Calcul automatique des déterminants. C. R. Acad. Sci. Paris 245 (1957), 824–826.

The author considers the practical application of his method of "cycles" for finding determinants [same C. R. 239 (1954), 383–385; MR 16, 405] on automatic digital computers. A flow chart is given and discussed in some detail.

U. W. Hochstrasser (Lawrence, Kans.).

See also: Numerical Methods: Forsythe. **Biology and Sociology:** Le Lionnais. **Control Systems:** Feldbaum.

PROBABILITY

Nasr, Saad K. On the convergence, basic distributions and classifications of abstract random variables. Teor. Veroyatnost i Primenen. 2 (1957), 178–186. (Russian summary)

Fréchet [see, e.g., Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, 1954, Deutsch. Verlag Wiss., Berlin, 1956, pp. 23–28; MR 18, 646; this review contains further references] defined stochastic convergence of a sequence of random variables, X_n, with values in a metric space, $\mathfrak{X}$. Doss [Bull. Sci. Math. (2) 73 (1949), 48–72; MR 11, 190] extended the definition to a family of random variables, X_λ, where λ belongs to a set F; the definition is "after" a filter on F. [For the definitions of "filter", "écart", "separated uniform structure" see Bourbaki, Topologie générale, Actualités Sci. Ind., no. 1196, Hermann, Paris, 1953; MR 14, 1106.] The author now generalises the definition still further, to spaces $\mathfrak{X}$ that are not neces-

sarily metric, but have a separated uniform structure determined by a family of pseudo-metrics (écarts). Thus, the new definitions of the various forms of stochastic convergence, (convergence in probability, strong convergence in probability, and almost certain convergence) are "relative" to the family of pseudo-metrics as well as being "after" a filter on F.

By introducing appropriate classes of numerical-valued functions of an abstract random variable, the author defines "basic distribution functions" analogously to the definition given by Doob [Stochastic processes, Wiley, New York, 1953, p. 47; MR **15**, 445]. He shows that when $\mathfrak{X}$ is a locally compact space, the basic distribution functions of a random variable define its law of probability. He also uses the basic distributions to classify abstract random variables, analogously to Doob, loc. cit. chapter II. (The reference to Doob, chapter 111, on page 185, should be to chapter III.) *I. J. Good.*

Šidák, Zbyněk. On relations between strict sense and wide sense conditional expectations. Teor. Veroyatnost. i Primenen. **2** (1957), 283–288. (Russian summary)

Let (X, F, μ) be a probability space, and let GCF be a σ-algebra of sets. Let $\xi\{f|G\}$ be the conditional expectation (in the strict sense) of the random variable f relative to the σ-algebra G [See J. L. Doob, Stochastic processes, Wiley, New York, 1953; MR **15**, 445]. Let L_2 be the Hilbert space of all random variables f with $\int f^2 d\mu < \infty$. Let $\hat{\xi}\{f|\mathfrak{M}\}$ be the projection of L_2 on the closed linear manifold $\mathfrak{M}$ (also called a conditional expectation in the wide sense). The paper discusses the relationship between conditional expectations $\xi\{\cdot|G\}$ and projections $\hat{\xi}\{\cdot|\mathfrak{M}\}$. Among other results, it is proved that conditional expectations are the uniquely defined continuous (in the mean of degree 1) extensions from L_2 to L_1 of all measurable projections. This generalises a result of Bahadur [Proc. Amer. Math. Soc. **6** (1955), 565–570; MR **17**, 286].

If $\xi\{f|G_n\}$ tends almost certainly to $\xi\{f|G\}$ for each f in L_1, then $\hat{\xi}\{f|L_2(G_n)\}$ tends in mean square to $\hat{\xi}\{f|L_2(G)\}$ for each f in L_2, where $L_2(G)$ is the space of all G-measurable random variables f in L_2.

A transformation T of Hilbert space L_2 into itself is a conditional expectation if and only if T is self-adjoint, idempotent, constant-preserving, and if, finally, $T\{\max(T\{f\}, T\{g\})\} = \max(T\{f\}, T\{g\})$ whenever f and g are in L_2. Bahadur, loc. cit., and Moy [Pacific J. Math. **4** (1954), 47–63; MR **15**, 722] proved similar results, but used the product fg instead of $\max(f, g)$. This gave rise to some difficulties because fg is not necessarily in L_2 when f and g are. *I. J. Good* (Cheltenham).

Driml, Miloslav; et Hanš, Otto. Sur la convergence presque sûre d'une suite d'éléments aléatoires de type L^*. C. R. Acad. Sci. Paris **246** (1958), 539–540.

Conditions for the (weak star or strong) convergence with probability one of functions from a probability measure space to the dual of a separable Banach space B in terms of a (countable) separating subset A of B.

H. P. McKean, Jr. (Cambridge, Mass.).

Paszkowski, S. Sur un problème du calcul de probabilité. II. Mouvement d'une molécule sur plusieures droites parallèles. Studia Math. **15** (1956), 273–299.

[For part I see Mycielski and Paszkowski, Studia Math. **15** (1956), 188–200; MR **19**, 588.] Let $\Pi_1, \Pi_2, \cdots, \Pi_n$ be parallel lines each provided with a measure. On this system a molecule moves in such a way that it can change the direction of its progress, jump from one line to another (in a direction perpendicular to the lines) or be stopped. Let $I_i \subset \Pi_i$ be segments which are orthogonal projections one on the other. If the molecule has just entered I_i from the left, p_{ij} denotes the probability that it emerges from the jth segment at the right, q_{ij} the probability that it emerges from the jth segment at the left, and r_{ij}, that it stops on the jth segment. Similarly, P_{ij}, Q_{ij} and R_{ij} denote the probabilities of transmission, reflection and absorption for incidence from the right. The author obtains six nonlinear functional equations {which the reviewer feels can be written somewhat more simply in terms of the matrices $p=(p_{ij})$, $q=(q_{ij})$, $\cdots$. For example, (97) is $p(x+y)=p(y)[1-Q(x)q(y)]^{-1}p(x)$ and similarly for the others}.

Assuming that the mass distributions are all equal, the author gets six differential equations. These are solved by ingenious transformations under the further hypothesis that $p_{ij}, \cdots$ has one value for $i \neq j$, another for $i=j$. The similarity to the case $n=1$ is explained by saying that the transformations refer to the "shadow" of the molecule, that is, to its projection, or another parallel line II.

{The problem is especially interesting because the same functional equations arise in transmission-line theory. With this observation the reviewer generalized the results for $n=1$ to the case of arbitrary dependence on the common end-points (x, y) of the I_j [Math. Ann. **133** (1957), 235–250; MR **19**, 655]. Those results have been generalized to arbitrary n by Reid [Solutions of a Riccati Matrix differential equation, to appear] and by the reviewer [Inequalities for a matrix Riccati equation, to appear]. Pasckowski remarks that he does not know whether his inequalities (120), (121) guarantee that the probabilities will be non-negative. The results in the papers just mentioned indicate that the answer is affirmative, and enable one to solve the general case of arbitrary dependence on the end-points (x, y). A typical one of the six matrix equations is: $p(x, z)=p(y, z)[1-Q(x, y)q(y, z)]^{-1}p(x, y)$. Passing to the limit yields a matrix Ricatti differential system which reduces, when the coefficients are constant, to the equations in the paper under review.} *R. M. Redheffer.*

Molčanov, A. M. Finite sets and the scalar product. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 920–922. (Russian)

Isii, Keiiti. Some investigations of the relation between distribution functions and their moments. Ann. Inst. Statist. Math., Tokyo **9** (1957), 1–11.

Three theorems are proved concerning the relation between two distribution functions whose first N moments are equal. For instance, if the moduli of continuity of two density functions $f(x)$ and $g(x)$, continuous over $[0, 1]$, are dominated by $\omega(\delta)$, and if $\int_0^1 x^r f(x)dx = \int_0^1 x^r g(x)dx$ for $r=1, \cdots, 2n-1$, then $\int_0^1 |f(x)-g(x)|dx < 5\omega(1/2n)$.

H. S. A. Potter (Aberdeen).

★ **Dugué, D. Arithmétique des lois de probabilités.** Mémor. Sci. Math., no. 137. Gauthier-Villars, Paris, 1956, 51 pp. 1000 francs.

This is a survey of properties of characteristic functions (ch.f.'s) including contributions by the author.

Chap. I is concerned with the reciprocity and continuity theorems (P. Lévy), conditions for analyticity (P. Lévy,

Dugué), and a relation between ch.f.'s and their Fourier series (Dugué). Chap. II is devoted to necessary and sufficient conditions for a function to be a ch. f. (Mathias and Bochner, Khintchine) and to sufficient conditions (Polyà and Titchmarsh, Dugué and Girault, Marcinkiewicz). Chap. III is concerned with the decomposability theorems (Lévy and Cramér, Raïkov), analyticity properties of the components and infinitely decomposable ch. f.'s (Lévy, Khintchine, Lukacs and Dugué). Chap. IV is devoted to various cases of indecomposability, and in the very short Chap. V are stated some properties relative to products of random variables. *M. Loève* (Berkeley, Calif.).

Zinger, A. A.; and Linnik, Iu. V. On a class of differential equations with an application to certain questions of regression theory. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 121–130. (Russian. English summary)

The relation $E(P/Q)=EP$, where P is a polynomial in independent and identically distributed random variables $X_1, \cdots, X_n$ with common characteristic function f and finite moments up to a suitable order and $Q=X_1+\cdots+X_n$, is transformed into a differential equation in f. It is shown that a positive-definite solution in a neighborhood of the origin can be extended to an entire function, and the normal case solution is considered. The result is applied to the determination of the distribution for which the regression line of P on $\bar{X}$ is horizontal. *M. Loève.*

Slepian, D. Fluctuations of random noise power. Bell System Tech. J. **37** (1958), 163–184.

In this paper the average power of a sample of finite time duration T of a Gaussian noise is investigated; its exact probability function and cumulative distribution function are computed for a number of ergodic noises and for a number of values of T, and presented as a series of curves. The evaluation of the probability density function

$$g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixz}\prod_{\nu=1}^{n}(1-2iz\lambda_\nu)^{-\frac{1}{2}}dz$$

presents difficulties even with high-speed computing equipment and is accomplished by a new method; this consists of deforming the contour of integration into a set of circles enclosing pairs of branch points $z_\nu=-i/(2\lambda_\nu)$, collapsing the circles and obtaining for $g(x)$ an alternating sum of finite integrals with integrands whose singularities at the end points of the intervals can be removed by a change of variables. *W. F. Freiberger.*

Dostupov, B. G.; and Pougatchev, V. S. The equation defining the probability distribution of the integral of a system of ordinary differential equations with random parameters. Avtomat. i Telemeh. **18** (1957), 620–630. (Russian. English summary)

The equations under discussion are of the general form

$$\frac{dY_\lambda}{dt}=f_\lambda(t, Y_1, \cdots, Y_n, v_1, \cdots, v_m) \quad (\lambda=1, \cdots, n),$$

where the f_λ are real-valued functions such that the $\partial f_\lambda/\partial Y_\mu$ are bounded and piecewise continuous, and the v_ν are uncorrelated random variables. Let p_t denote the joint probability density of $Y_1, \cdots, Y_n, v_1, \cdots, v_m$ at time t. The authors show that p_t satisfies the partial differential equation $\partial p_t/\partial t+\sum_{\mu=1}^{n}\partial(p_t f_\mu)/\partial y_\mu=0$, and obtain an approximate expression for p_t in terms of p_0.

L. A. Zadeh (New York, N.Y.).

Konijn, H. S. A class of two-dimensional random variables and distribution functions. Sankhyā **18** (1957), 167–172.

Various properties of the covariance are given for the class of bivariate random variables generated by a rotation of independent random variables through an angle θ. Some of the results are related to the characterization of the normal d.f. in terms of the independence of linear forms. *I. Olkin* (Stanford, Calif.).

Shapiro, J. M. Sums of independent truncated random variables. Ann. Math. Statist. **28** (1957), 754–761.

Let $\{x_{n,k}\}$ $(k=1, 2, \cdots, k_n; n=1, 2, \cdots)$ be a double sequence of infinitesimal random variables, with $x_{n,1}, \cdots, x_{n,k_n}$ independently distributed for each n. For $a>0$ let

$$x_{n,k}{}^a=\begin{cases} x_{n,k}, & \text{if } -a<x_{n,k}\leqq a,\\ 0, & \text{otherwise.}\end{cases}$$

Let $F_n(x)$ be the distribution function of $x_{n,1}+\cdots+x_{n,k_n}$ and let $F_n{}^a(x)$ be the distribution function of $x_{n,1}{}^a+\cdots+x_{n,k_n}{}^a$. For any infinitely divisible distribution function $F(x)$ let $\varphi(t)$ be its characteristic function in the Khintchine form

$$\varphi(t)=\exp\Big\{iyt+\int_{-\infty}^{\infty}\Big[e^{iut}-1-\frac{iut}{1+u^2}\Big]\frac{1+u^2}{u^2}dG(u)\Big\}.$$

The following are the main results: I) If $F_n(x)$ converges to $F(x)$ and if $\pm a$ are continuity points of $G(u)$, then $F_n{}^a(x)$ converges to $F(x)$ if and only if $G(u)$ is constant for $u\leqq -a$ and also for $u\geqq a$. II) If $F_n{}^a(x)$ converges to $F(x)$, then $F(x)$ has finite moments of all orders, and the kth moment of $F_n{}^a(x)$ converges to the kth moment of $F(x)$. *J. R. Blum* (Bloomington, Ind.).

Rihter, Vol'fgang. Lokale Grenzwertsätze für grosse Abweichungen. Teor. Veroyatnost i Primenen. **2** (1957), 214–229. (Russian. German summary)

Let $\{X_j\}$ $(j=1, 2, \cdots)$ be a sequence of independent random variables with distribution functions $V_j(x)$, and let there exist $DX_j=\sigma_j{}^2$, $s_n{}^2=\sum_{j=1}^{n}\sigma_j{}^2$, $EX_j=0$ for $j=1, 2, \cdots$. Set $Z_n=\sum_{j=1}^{n}X_j/s_n$. With the help of the saddle-point method in function theory the author derives several local limit theorems that are analogous to known limit theorems for large deviations due to Cramér [Confér. Internat. Sci. Math., partie III, Hermann, Paris, 1938, pp. 5–23] and Petrov [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 4(62), 195–202; Vestnik Leningrad Univ. **8** (1953), no. 8, 13–25; MR **16**, 378; **17**, 979]. The latter considered the behaviour of $P(Z_n<x)=F_n(x)$ for $n\to\infty$ where, however, x also tended to infinity with n. Petrov generalized Cramér's theorem for the case of identically distributed X_j to the general case and also improved the remainder term and growth with x. The author proves as Theorem 1: Let conditions A through C be satisfied. Then for sufficiently large n the random variable Z_n possesses a density function $p_{Z_n}(x)$. Moreover, let $x>1$ and $x=o(n^{\frac{1}{2}})$ for $n\to\infty$. Then $p_{Z_n}(x)/\phi(x)=[1+O(xn^{-\frac{1}{2}})]\exp[xn^{-\frac{1}{2}}\lambda_n(xn^{-\frac{1}{2}})]$, where $\lambda_n(t)$ is a power series which for sufficiently small values of $|t|$ converges uniformly for all n and $\phi(x)$ is the normal density function. For negative x an analogous relation holds. In Theorem 2 for the case of identically distributed X_j, the author shows how condition C can be weakened, and in Theorem 3 the author gives an analogous limit relation for the case of lattice random variables X_j.

H. P. Edmundson (Santa Monica, Calif.).

Kendall, David G. A note on Doeblin's central limit theorem. Proc. Amer. Math. Soc. **8** (1957), 1037–1039.

In this note, the author shows by means of delicate arguments that the central limit theorem of Doeblin [Bull. Soc. Math. France **66** (1938), 210–220] remains valid, even if the condition that the recurrence-times of the Markov chain considered by Doeblin has finite second moments is left out. For the asymptotically normal distribution of the sums $S_n=\sum_{m=0}^{n} f(X_m)$ of the statistic variables $f(X_m)$, determined by the state X_m of a Markov chain with a countable infinity of states $j=1, 2, \cdots$ forming a single positive recurrent class, it is sufficient that the mean recurrence times m_{jj} is finite, that the partial sums obtained by adding terms for different recurrence periods will have a finite mean value $\mu_j=Mm_{jj}$ and a finite variance $\sigma_j^2=Bm_{jj}$. The asymptotically normal distribution in question will have the mean value nM and the variance nB. *L. Törnqvist* (Helsinki).

Barra, Jean-René. Du comportement de la moyenne d'un échantillon. C. R. Acad. Sci. Paris **244** (1957), 2002–2004.

Let $S_n=(1/n)\sum_{i=1}^{n} X_i$ be the mean of a random sample from a random variable X with characteristic function $\varphi(t)$. The author shows that a necessary and sufficient condition for S_n (not further normalized) to have a limiting distribution as n becomes infinite is that $\varphi(t)$ have a right (or left) derivative at the origin. The limiting distribution is then Cauchy. *M. D. Donsker.*

Teicher, Henry. On the convergence of projected distributions. Ann. Inst. Statist. Math., Tokyo **9** (1958), 79–86.

Let X_n, Y_n $(n=1, 2, \cdots)$ be real random variables with c.d.f.'s $L(X_n)$, $L(Y_n)$ and c.f.'s $\phi_{X_n}(t)$, $\phi_{Y_n}(t)$ respectively. Suppose that there exist c.d.f.'s $F(x)$, $G(x)$ with corresponding c.f.'s $f(t)$, $g(t)$ such that $L(X_n)\to F$, $L(Y_n)\to G$. The author obtains several sets of necessary and sufficient conditions that $L(cX_n+Y_n)\to F(x/c)*G(x)$, where $c\neq 0$ and $*$ denotes convolution. Analogous results are noted for the case of a sequence of k dimensional vectors $X_n=X_{n_1}, X_{n_2}, \cdots, X_{n_k}$ where it is known that $L(X_{n_j})\to F_j$ $(j=1, \cdots, k)$ and it is desired that $L(\sum_{j=1}^{k} c_j X_{n_j})\to F_1(x/c_1)*F_2(x/c_2)*\cdots*F_k(x/c_k)$. Several examples are given. *H. P. Edmundson* (Santa Monica, Calif).

Rozanov, Yu. A. On a local limit theorem for lattice distributions. Teor. Veroyatnost. i Primenen. **2** (1957), 275–281. (Russian. English summary)

Let (1) $\zeta_1, \zeta_2, \cdots$ be a sequence of mutually independent, integer valued random variables such that $a_k=E\zeta_k$ and $b_k^2=E(\zeta_k-a_k)^2$ exist. Let $S_n=\zeta_1+\cdots+\zeta_n$, $A_n=ES_n$, $B_n^2=E(S_n-A_n)^2$. It is assumed that $B_n\to\infty$ as $n\to\infty$ and

$$b_k^{-2}\sum_{|j-a_k|\leq N}(j-a_k)^2 p_{kj}\to 1$$

as $N\to\infty$, uniformly in k. The following is proved: In order that

$$P\{S_n=m\}=(2\pi)^{-1}B_n^{-1}\exp\{-(m-A_n)^2/(2B_n^2)\}+o(B_n^{-1})$$

as $N\to\infty$, uniformly in m $(m=0, \pm 1, \pm 2, \cdots)$, and that the same be true for any sequence like (1) which differs from (1) only by a finite number of terms, it is necessary and sufficient that

$$\prod_{k=1}^{\infty}\max_{0\leq m<h} P\{\zeta_k\equiv m(\mathrm{mod}\ h)\}=0,\quad h\geqq 2.$$

W. Hoeffding (Chapel Hill, N.C.).

Bühlmann, Hans. Le problème "limite central" pour les variables aléatoires échangeables. C. R. Acad. Sci. Paris **246** (1958), 534–536.

The author states some interesting results about the asymptotic behaviour of normed sums of exchangeable random variables $X_1, X_2, \cdots$. For example, when the expectations vanish and the variances are finite, the limit laws are weighted normal; and they are ordinary normal if and only if

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} 0,\quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} 0.$$

The proofs and other results are to appear in the University of California Series in Statistics. *M. Loève.*

Bergström, Harald. On the limit theorems for convolutions of distribution functions. I. An analysis in the Weierstrass norm. J. Reine Angew. Math. **198** (1957), 121–142.

This is the first part of a study of limit theorems in probability theory. One of the topologies used is the Weierstrass norm $N_\sigma[f(x)]$ of a function $f(x)$

$$N_\sigma[f(x)]=\sup_x |f(x)*\Phi(x/\sigma)$$

where $\Phi(x)$ is the normal distribution function. To emphasize that his method is elementary, the author has made his paper self-contained and introduces in the first few sections concepts like distribution functions, Riemann-Stieltjes integrals, weak convergence and convolutions. Quasi-distribution functions are defined as functions which are finite and non-decreasing for $x\neq 0$.

This part of the study develops results which will be needed later on and contains a series of lemmas on the properties of the Weierstrass norm. These are used to study the convergence of a sequence of quasi-distribution functions. The main result is the following theorem: Consider a sequence $f_n(x)$ of bounded quasi-distribution functions tending to zero as $x\to\pm\infty$. This sequence converges in the Weierstrass norm if and only if 1) $f_n(x)$ converges weakly to a quasi-distribution function $f(x)$ with $f(\pm\infty)=0$, and 2) the truncated moments $\int_{|x|\leq\eta} x^\nu df_n(x)$ $(\nu=1, 2)$ tend to finite values as $n\to\infty$ for every η such that $\pm\eta$ are continuity points of $f(x)$.

Certain limit theorems are stated in the introduction; they will be proved in the forthcoming second part of this paper. *U. Grenander* (Providence, R.I.).

Sevast'yanov, B. A. Limit theorems for branching stochastic processes of special form. Teor. Veroyatnost. i Primenen. **2** (1957), 339–348. (Russian. English summary)

Betrachtet wird ein altersunabhängiger verzweigter Prozeß, bei dem jedes bereits vorhandene Teilchen innerhalb der Zeit Δt mit der Wahrscheinlichkeit $\delta_{k1}+p_k\Delta t+o(\Delta t)$, wenn $\Delta t\to 0$, k weitere Teilchen erzeugt, unabhängig von allen anderen Teilchen, und bei dem außerdem innerhalb der Zeit Δt mit der Wahrscheinlichkeit $\delta_{k0}+q_k\Delta t+o(\Delta t)$ noch k neue Teilchen entstehen. Dabei ist $\sum_{k=0}^{\infty} p_k=\sum_{k=0}^{\infty} q_k=0$ und δ_{kl} das Kroneckersche Symbol. Es sei $f(x)=\sum_{k=0}^{\infty} p_k x^k$, $g(x)=\sum_{k=0}^{\infty} q_k x^k$, $a_1=f'(1)$, $b_1=f''(1)$, $c_1=f'''(1)$, $a_2=g'(1)$ und $b_2=g''(1)$. Untersucht wird das asymptotische Verhalten der Anzahl μ_t der Teilchen zur Zeit t unter der Annahme $\mu_0=0$ in den drei Fällen $a_1<0$, $a_1=0$ und $a_1>0$. Im ersten Fall existieren, falls $a_2<+\infty$, die Grenzwerte $P_k=\lim_{t\to\infty} P\{\mu_t=k\}$ und es ist $\sum_{k=0}^{\infty} P_k x^k=\exp\{\int_x^1 (g(y)/f(y))dy\}$. Im zweiten

Fall existiert, falls b_1, c_1, a_2 und b_2 endlich und b_1 und a_2 positiv sind, der Grenzwert $\lim_{t\to\infty} \mathbf{P}\{\mu_t/a_2 t<y\}$ für jedes y und kann explizit angegeben werden. Im dritten Fall schließlich existiert, falls b_1, a_2 und b_2 endlich sind, $\lim_{t\to\infty} \mathbf{P}\{\mu_t/e^{a_1 t}<y\}=S(y)$, und die charakteristische Funktion von S wird explizit angegeben.

K. Krickeberg (Hamburg).

★**Fortet, Robert M. Les fonctions aléatoires comme éléments aléatoires dans un espace de Banach.** Proceedings of the International Congress of Mathematicians 1954, Amsterdam, vol. III, pp. 356–359. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Let B be a Banach space with elements x, and B^* its dual with elements x^*. Let θ denote the null element of B. The author considers $z_n=n^{-1}\sum_{i=1}^{n} X_i$, where $X_1, X_2, \cdots$, is a sequence of independent identically distributed random elements in B with $E(X_i)=\theta$, $E(\|X_i\|^2)<\infty$. If $\varphi_n(x^*)$ is the characteristic function of z_n, then $\varphi_n(x^*)$ tending toward a limit $\varphi(x^*)$ does not in general imply $\varphi(x^*)$ is a characteristic function. However, the author gives conditions on B which imply that $\varphi(x^*)$ is the characteristic function of a random Laplacian (Gaussian) element y of B, with $E(y)=\theta$, $E(\|y\|^2)<\infty$. Under further assumptions on B, this last implies that the distribution of z_n tends to that of y. These results give a more general approach to the convergence of distributions of functionals on processes. *M. D. Donsker* (Minneapolis, Minn.).

Volkonskiĭ, V. A. A multidimensional limit theorem for Markov chains with a countable set of states. Teor. Veroyatnost i Primenen. **2** (1957), 230–255. (Russian. English summary)

Consider a denumerable Markov chain with stationary transition probabilities which forms one recurrent class. Let N_n be the $(r+1)$st dimensional vector whose sth component is the number of visits to state number s in the first n steps, $s=0, 1, \cdots, r$. Assume that the recurrence time distribution of the state number 0 satisfies the Doeblin condition for belonging to the domain of attraction of the stable law with exponent α, $0\leq\alpha<2$, $\alpha\neq1$. Does this imply that every state has the same property? This question does not seem to be discussed [for analogous considerations see Chung, Trans. Amer. Math. Soc. **76** (1954), 397–419, Th. 3 and footnote to it; MR **16**, 149]. Under the above assumption it is shown that by a suitable affine norming, the distribution of $A_nN_n+B_n$ converges to a nondegenerate distribution which is explicitly given. *K. L. Chung.*

Jacobs, Konrad. Zur Theorie der Markoffschen Prozesse. Math. Ann. **133** (1957), 375–399.

The author applies his decomposition theorem [Math. Z. **64** (1956), 298–338; cf. also Abh. Math. Sem. Univ. Hamburg **21** (1957), 194–246; MR **17**, 988; **19**, 778] to Markov processes. The finite-dimensional case is first given as an illustration of the method, this being one of the methods presented in the second cited paper. Next he takes up the general case previously treated by Kryloff-Bogoliouboff, Doeblin-Doob and Yosida-Kakutani. The assumption is: the weak closure of the semigroup contains a weakly complete transformation. It is shown that the strong version of this assumption is weaker than that of Kryloff-Bogoliouboff, which is in turn weaker than that of Doeblin-Doob. It is stated that Yosida-Kakutani has shown the latter implication under denumerability assumptions which are superfluous. The present method is said to lean essentially on a theorem by Eberlein on weak compactness and the use of filters.

K. L. Chung (Syracuse, N.Y.).

Waugh, W. A. O'N. Conditioned Markov processes. Biometrika **45** (1958), 241–249.

The author considers temporally homogeneous Markov chains with a continuous time parameter and non-negative integer states. Let C be a positive class of states (C is maximal closed, its states are communicating, and each state in it has a finite mean recurrence time). Let A be the event that the state eventually is in C, $u_i=P(A|\text{initial state is } i)$, and let F be the set of states for which $u_i>0$. If $p_{ij}(t)$ is the transition matrix for the original chain, define a new matrix by $\tilde{p}_{ij}(t)=(u_j/u_i)p_{ij}(t)$, $i\in F$; and $\tilde{p}_{ij}(t)=p_{ij}(t)$, $i\notin F$. Also define new initial probabilities by $\tilde{p}_i=p_iu_i/\sum_{j=0}^{\infty} p_ju_j$, where the p_i are the initial probabilities for the original chain. Then the new matrix (which determines probabilities denoted by $\tilde{P}$) has the property that for any event E, $P(E|A)=\tilde{P}(E)$. Example: consider a birth-and-death process with birth rate$=\lambda>\mu=$death rate. Then the conditional probability of any event, given that extinction occurs, equals the unconditional probability of the same event determined for a birth-and-death process with birth rate μ and death rate λ. More general birth-and-death processes are considered, and also the Galton-Watson family-tree model [the result for this case was given by Otter, Ann. Math. Statist. **20** (1949), 206–224 (see p. 218); MR **11**, 41].

T. E. Harris (Santa Monica, Calif.).

Mauldon, J. G. On non-dissipative Markov chains. Proc. Cambridge Philos. Soc. **53** (1957), 825–835.

Let $P=(p_{ij})=(p_{ij}^{(0)})$ be a Markov chain with an enumerably infinite number of states. As is well-known, the time average $\Pi_{ij}=\lim_{n\to\infty} q_{ij}^{(n)}$ exists ($q_{ij}^{(n)}=n^{-1}\sum_{k=0}^{n-1} p_{ij}^{(k)}$, $p_{ij}^{(k)}=\sum_{l=1}^{\infty} p_{il}^{(k-1)}p_{lj}$) and satisfies the conditions $\Pi_{ij}\geq0$ and $\sum_{j=1}^{\infty}\Pi_{ij}\leq1$. D. G. Kendall [same Proc. **47** (1951), 633–634; MR **13**, 51] proved that P is non-dissipative, i.e., $\sum_{j=1}^{\infty}\Pi_{ij}=1$ for $i=1, 2, \cdots$, if there exists a non-negative vector $w=(w_1, w_2, \cdots)$ such that $\lim_{i\to\infty} w_i=\infty$ and $c_i=\sum_{j=1}^{\infty} p_{ij}w_j-w_i\leq0$ for all i. The present author extends Kendall's result as follows: P is non-dissipative if there exists a non-negative vector $w=(w_1, w_2, \cdots)$ such that $c_i=\sum_{j=1}^{\infty} p_{ij}w_j-w_i<\infty$ whenever the state i is a null state, and $\lim_{i\to\infty}\sup c_i<0$. Here, the state i is, for P, a null state if it is recurrent with infinite mean recurrence time [cf. W. Feller, An introduction to probability theory and its applications, v. 1, Wiley, New York, 1950; MR **12**, 424]. *K. Yosida.*

Hunt, G. A. Markoff processes and potentials. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 414–418.

In this short note the author outlines without proofs a generalization of the well known relation of Brownian motion to potentials. He considers a certain class of stationary Markoff processes and, for them, introduces the concepts of right and left potential of a measure, right and left regular points of an analytic set E, the capacity of E, etc. For this class of processes, and with these generalized notions of capacity and regular point, he gives several equivalent formulations of the statement: if F is compact and has strictly positive capacity, then some point is right regular for F. *M. D. Donsker.*

Goodman, Leo A. Simplified runs tests and likelihood ratio tests for Markoff chains. Biometrika **45** (1958), 181–197.

Consider a long s-state Markov chain with discrete time. The author considers several statistics that are appropriate for testing various null hypotheses within or against various non-null hypotheses. He first considers the case $s=2$, for which the number of runs (i.e. of consecutive occurrences of a single state) is the basic statistic for distinguishing between randomness and first-order Markovity, and he discusses and extends the work of F. N. David [Biometrika **34** (1947), 299–310; MR **9**, 600] and of P. G. Moore [ibid. **40** (1953), 111–115; MR **14**, 1104]. Then, for $s>2$, he again considers hypotheses for which statistics based on runs are appropriate. Finally he makes some corrections and extensions of a paper by the reviewer [ibid. **42** (1955), 531–533; **44** (1957), 301; MR **17**, 381] in which relationships were pointed out between likelihood-ratio statistics and quadratic statistics ('psi-squared') for testing Markovity of one order (including order -1, i.e. 'perfect' randomness) within Markovity of a higher order.

Slips in proof-reading occur at the following places: P. 182, section 2, last two lines (for 'probabilities' read 'numbers' and for 'now' read 'not'), and formula (1) (to correct this refer to formula (2)); page 183, line 6; page 191, section 7, line -4; page 192, line -9 (for 'where' read 'were'). *I. J. Good* (Cheltenham).

★ **Davenport, Wilbur B., Jr.; and Root, William L. An introduction to the theory of random signals and noise.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1958. ix+393 pp. $10.00.

The first part of this book is devoted to a discussion of the fundamental concepts and relations in probability theory which are used throughout the remainder of the book. Among the things covered are random variables and probability distributions, the central limit theorem, stationary processes, correlation functions and spectra. Special emphasis is given to the representation of a stochastic process by means of orthogonal functions.

The subject of the next chapter is shot noise, whose statistical properties are studied for the cases of temperature-limited and space charge-limited diodes. Probability distributions and spectral densities are derived. The partition noise in a pentode is also considered, while thermal noise is dealt with in a later chapter.

In this way one is led in a natural way to gaussian processes. The distributions are derived for the envelope and phase of a narrow band process and of a narrow band process plus sine wave.

The authors describe linear systems and their effect upon random input. Introducing various noise figures, they discuss the goodness of the performance of amplifiers with respect to the generated noise. The reader is shown how one can define optimality of smoothing and prediction filters, and how such optimal filters can be actually found. This is done very concisely by restricting the processes to have rational spectral densities, so that the factoring of the density can be done without difficulty.

The important subject of non-linear devices is dealt with in two chapters. The first of these uses a direct approach, while the second proceeds via a Laplace transform representation of the device. Many cases are treated; all of them assume that the device is instantaneous (i.e. has no memory).

Chapter 14 on the statistical detection of signals surveys the methods that have been suggested for this purpose. It is shown how the problem can be considered as an application of the theory of testing hypotheses and estimating parameters. This is carried out in a number of applications to radio and radar, and optimal detectors are derived.

The exposition is clear and the book reads very well. The authors have apparently tried to write a book with the emphasis on fundamental ideas rather than on detailed results, and they have been quite successful. This makes the book of general interest to communication engineers, applied mathematicians and mathematical statisticians. A large number of well chosen problems accompanies each chapter and further augments the value of this useful book. *U. Grenander.*

Lomnicki, Z. A.; and Zaremba, S. K. On some moments and distributions occurring in the theory of linear stochastic processes. I. Monatsh. Math. **61** (1957), 318–358.

The authors consider a linear stochastic process of the form $x_t=\sum_0^\infty h_i\varepsilon_{t-i}$. Put $R_k=E(x_tx_{t+k})$, $R_{k,N}=N^{-1}\sum_1^N x_tx_{t+k}$, $C_{k,N}=R_{|k|,N-|k|}$. The authors, paper culminates in the following result. If $\sum|h_i|$ converges and if $\{\varepsilon_i\}$ is stationary in the strict sense with finite fourth moment, the ε's being independent with 0 mean, then for any fixed set of integers $k(1), \cdots, k(l)$ the limiting distribution of each of the two sets of random variables $N^{\frac{1}{2}}(C_{k(v),N}-R_{k(v)})$ $(v=1, \cdots, l)$ and $N^{\frac{1}{2}}(R_{k(v),N}-R_{k(v)})$ when $N\to\infty$ is normal. The covariance matrix of this limit distribution is also computed by the authors. This result remains valid even if a trend is first eliminated and the covariances of the process estimated from the residuals. A similar result is established also for the estimators of the autocorrelation function. The results of the authors are also applied to establish the ultimate normality of various estimates important in the spectral analysis of the process. *H. B. Mann* (Columbus, Ohio).

Scheidegger, A. E. The random-walk model with autocorrelation of flow through porous media. Canad. J. Phys. **36** (1958), 649–658.

The usual random walk on the real line with no drift is considered with steps $\pm d$ occupying time intervals τ with the difference that the successive steps are not independent. Instead it is assumed that the probability that the particle will reverse direction after a step is $\frac{1}{2}(1-c)$, $c>0$, independent of all previous steps. The author shows that in the limiting case where $d\to0$, $\tau\to0$, $(1-c)/\tau\to1/A$ (so that $c\to1$) and $d/\tau\to v$ (contrary to the usual case with $c=0$ where $d^2/\tau\to v$) the probability density at x at time t satisfies the telegraph equation

$$\frac{\partial^2\psi}{\partial t^2}+\frac{1}{A}\cdot\frac{\partial\psi}{\partial t}=v^2\frac{\partial^2\psi}{\partial x^2}$$

instead of the diffusion equation obtaining when $c=0$. The constants, A and v, are related to other dynamical variables and the value of the inclusion of the correlation effect between successive steps is discussed.

D. V. Lindley (Cambridge, England).

★ **Doob, J. L. A new look at the first boundary-value problem.** Applied probability. Proceedings of Symposia in Applied Mathematics, Vol. VII, pp. 21–33. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

A probabilistic abstraction and extension is given of the

Perron-Wiener-Brelot method for solving the Dirichlet problem for harmonic functions. Let $R \cup R'$ be a compact metric space, where R is an open dense set of the space, with no points in common with the "boundary" R'. It is supposed that there is a distinguished class of open sets, called "regular sets", with the properties: the closure of a regular set is a subset of R, and every open subset D of R is the union of a monotone sequence of regular sets whose closures are subsets of D. It is further assumed that if D is a regular set, and if $z \in D$, then there exists a measure $\mu(z, D, \cdot)$ of Borel subsets of the boundary D' with $\mu(z, D, D')=1$ having the properties: (1) If f is a non-negative Borel measurable function defined on D', then the average $u(z)=\int_{D'} f(\zeta)\mu(z, D, d\zeta)$ is finite and continuous on an open set $D_1 \subseteq D$, when it is finite on a dense subset of D_1. (2) u is continuous on $D \cup D'$ with boundary value f if f is continuous. (3) The average is "regular" on any set on which it is finite — "regular" means that it is equal to its average. (4) Every function which is regular in a neighbourhood of each point of its domain of definition is regular.

The "Poisson" kernel $\mu(z, D, \cdot)$ enables us to define probabilistic trajectories to the boundary as follows. Let $\{R_n\}$ be a monotone sequence of regular sets, with union R such that the closure of R_n is $\subseteq R_{n+1}$. If z_0 is any point of R, let N be the first value of n for which $z_0 \in R_n$. Then choose $z_1=z_2=\cdots=z_{N-1}=z_0$, and choose z_N at random on R_N', with the distribution $\mu(z_0, R_N, \cdot)$; then choose z_{N+1} on R_{N+1}' with the distribution $\mu(z_N, R_{N+1}, \cdot)$, and so on. In this way, a random walk on R is defined which approaches R'. We denote by $R(\omega)$ the set of limit points of R' of the "ω-trajectory". Considering the regular functions as if they are harmonic functions, we may define sub- and super-regular functions by Perron method. Thus the lower [upper] class of functions, for a given boundary function f on R', is defined as the class of functions on R, including the function $-\infty$ [∞] and every sub-regular [super-regular] function satisfying

$$\lim_{n\to\infty} u(z_n(\omega)) \leqq \inf_{\zeta \in R(\omega)} f(\zeta) \quad [\lim_{n\to\infty} u(z_n(\omega)) \geqq \sup_{\zeta \in R(\omega)} f(\zeta)].$$

Starting from these classes, the P-W-B method for solving the Dirichlet problem for harmonic functions is extended for regular functions. For this extension, the fact that the sequence $\{u(z_n(\omega))\}$ is a semimartingale or martingale, for u sub-regular or regular, respectively, plays an important role. It is stressed that the introduction of the above probabilistic approaches to the boundary is much more intrinsic than the radial or non-tangential approaches to the boundary. *K. Yosida.*

Čerkasov, I. D. On the equations of Kolmogorov. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 237–244. (Russian)

The two equations of diffusion:

$$\partial f(s, x, \xi)/\partial s = a(x)(\partial^2 f(s, x, \xi)/\partial x^2) + b(x)(\partial f(s, x, \xi)/\partial x)$$

$$\partial f(s, x, \xi)/\partial s = \partial^2(a(\xi)f(s, x, \xi))/\partial \xi^2 - \partial(b(\xi)f(s, x, \xi))/\partial \xi$$

have a unique solution $f(s, x, \xi)$, which is a transition density function satisfying the usual Lindeberg-type mean and variance conditions if: (1) $a(x)>0$, $[a(x)]^{-1}$, $b(x)-2^{-1}a'(x)$ and $b'(x)-2^{-1}a''(x)$ are bounded for $x \in (-\infty, \infty)$; (2) $a''(x)$ and $b'(x)$ exist and are continuous for every x. These conditions are weaker than those of Feller, which require, e.g., that $a''(x)$ and $b'(x)$ are Lipschitzian. The paper by V. Feller, in Russian, cited in the "cited Literature" seems to be the same as W. Feller, Math. Ann. 113 (1936), 113–160. *K. L. Chung* (Syracuse, N.Y.).

Batschelet, Eduard. Zur Theorie der wiederkehrenden Ereignisse. I. Arch. Math. 8 (1957), 184–191.

Let $\{x_n\}$ $(n=1, 2, \cdots)$ be a sequence of independent random variables assuming the values one or zero with probability p and $1-p$ respectively. Let $H=z/t$ be a rational number with $0<H<1$ and z and t relatively prime, and let $\bar{x}_\nu=\nu^{-1}\sum_{i=1}^{\nu} x_i$. Then the event $\bar{x}_\nu=H$ is a recurrent event of period t. The author derives a recursion formula for $w(m, k)$, the probability that in $n=mt$ trials the event $\bar{x}_\nu=H$ occurs exactly k times. In the case $p=H=\frac{1}{2}$, this yields the recursion formula $w(m, k)=(1/2^{2m-k})\binom{2m-k}{m}$. He shows further that the event $\bar{x}_\nu=H$ is certain(persistent) if $p=H$ and that otherwise it is uncertain (transient). *J. R. Blum* (Bloomington, Ind.).

Batschelet, Eduard. Zur Theorie der wiederkehrenden Ereignisse. II. Arch. Math. 8 (1957), 294–297.

In the notation of the paper reviewed above the author defines an oscillation as the event $\bar{x}_{\nu-1}-p<0$, $\bar{x}_{\nu+1}-p>0$ or $\bar{x}_{\nu-1}-p>0$, $\bar{x}_{\nu+1}-p<0$ .The main result is that in the case $p=\frac{1}{2}$, the probability that in $2m$ trials exactly k oscillations occur is given by $(1/2^{2m})\binom{2m+1}{m-k}$.

J. R. Blum (Bloomington, Ind.).

Senju, Shizuo. The optimal assignment of operatives in serving automatic machines. Proc. Fac. Engrg. Keio Univ. 9 (1956), 81–91.

This is an extension of the work of C. Palm [Industritidningen Norden 75 (1947), 75–80, 90–94, 119–123] for a repair operative servicing k identical machines, each having an exponentially distributed working life till failure. The repair time is distributed according to a general gamma (Erlang-ν type) distribution. Tables are given for machine availability and service utilization for $\nu=2$. Results for other values of ν may be obtained by interpolation between this, $\nu=1$ (exponential) and $\nu=\infty$ (constant service time), both of which are known. Curves are given for machine assignments for different situations, to optimize productivity. *P. M. Morse.*

White, Harrison; and Christie, Lee S. Queuing with preemptive priorities or with breakdown. Operations Res. 6 (1958), 79–95.

Preemptive priority is defined and contrasted with head-of-the line priority. Poisson arrivals at a single server are assumed throughout, and only steady-state solutions are found. The effect on service-time statistics of preempted items re-entering service according to various rules is considered. A formula for the joint probability distribution of queue lengths of two preemptive priority classes is derived assuming negative exponential service statistics. Moments of this queue length distribution are found by a generating function approach. For any number of preemptive priority classes, expected times-in-system are found for a general service time distribution for each class, and for negative exponential service, with the same rate for each class, the generating functions of the delay distributions are also obtained. The similarity of queuing with breakdown to queuing with preemptive priority is pointed out, and two models of breakdown effects are worked out. [Authors' summary.]

P. M. Morse (Cambridge, Mass.).

★ **Jogdeo, S. S.; and Chaudhuri, A. K. Estimation of most economic number of repairmen required for a set of machines.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 237–252. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

Suppose one has m identical machines and r repairmen of equal skill, with the life times and repair times exponentially distributed. It is assumed that the cost per unit time of idle machines is proportional to the average number of idle machines and the cost per unit time of repairmen is proportional to the number of repairmen. The problem is to calculate the value of r for a given set of machines that yields a minimum total cost for failures and repairs. A derivation of the formulas which determine this optimum value of r is followed by a discussion of the best way to construct tables of cost as a function of r.

G. Newell (Stockholm).

Mack, C.; Murphy, T.; and Webb, N. L. The efficiency of N machines uni-directionally patrolled by one operative when walking time and repair times are constants. J. Roy. Statist. Soc. Ser. B. **19** (1957), 166–172.

Mack, C. The efficiency of N machines uni-directionally patrolled by one operative when walking time is constant and repair times are variable. J. Roy. Statist. Soc. Ser. B. **19** (1957), 173–178.

One man patrols N machines cyclically. He takes time k_i to walk to the ith machine from the previous one, inspect it, and service it routinely. If a machine has stopped, he takes an additional (possibly random) time c to repair it. The machines break down independently; the probability that a machine is still running at a time t after service or repair is e^{-t}. The distribution of c is the same for all i and independent of time to breakdown.

The process is a finite (if c has only finitely many possible values), aperiodic, irreducible Markov process in which a state is a round, specified by repair time required for each machine. From the steady-state probabilities, the average time per round is obtained, and thence the fraction of time spent on repair and the fraction of time each machine is running. Explicit formulas and a table are given in the case of nonrandom c (first paper). In the case of random c (second paper), the problem is reduced to the solution of linear equations.

In obtaining the steady-state probabilities, the k_i may be taken equal, since the transition probabilities depend on them only through their sum. Then, by symmetry, the probabilities don't depend on the time at which a round begins. This leads to equations in the probabilities. (The authors actually use similar equations for the marginal probabilities for machines 1, $\cdots$, $N-1$.) These are not the equations referred to by the uniqueness theorem for finite, irreducible, aperiodic Markov chains; therefore, the reviewer believes that the authors' appeal to it is invalid. However, they give another uniqueness proof for the case of random c. For non-random c it fails, but a similar proof works.

J. W. Pratt (Cambridge, Mass.).

Čistyakov, V. P. Local limit theorems for branching processes. Teor. Veroyatnost. i Primenen. **2** (1957), 360–374. (Russian. English summary)

Let μ_t be the random function of a continuous-time branching process, starting with one initial particle. The process is described by constants $p_k=\lim_{t\to 0}[P\{\mu_t=k\}/t]$, $k\neq 1$, and $p_1=\lim_{t\to 0}[(1-P\{\mu_t=1\})/t]$. Let $f(x)=\sum p_k x^k$, $a=f'(1)$, $b=f''(1)$, $c=f'''(1)$, $d=f^{\mathrm{iv}}(1)$, $P_n(t)=P\{\mu_t=n\}$, $P_n^*(t)=P\{\mu_t=n|\mu_t>0\}$. Theorem 1: Suppose $a=0$, and b, c, $d<\infty$. Let $z_{n,t}=2n/bt$. Then if $t\to\infty$ such that $0<c_1\leq z_{n,t}\leq c_2<\infty$, we have $(bt/2)P_n^*(t)=\exp(-z_{n,t})+O[(\ln t/t)^{\frac{1}{2}}]$. Theorem 3: Suppose $a>0$, $b<\infty$, and suppose h is the greatest common divisor for twin differences of indexes n for which $p_n\neq 0$, λ is the smallest nonnegative root of $f(x)=0$, $z_{n,t}=ne^{-at}(1-\lambda)$, $s(y)$ is the probability density corresponding to the limiting distribution $S(y)=\lim_{t\to\infty} P\{\mu_t(1-\lambda)e^{-at}<y|\mu_t>0\}$. Then if $t\to\infty$ so that $0<c_1\leq z_{n,t}\leq c_2$, we have $e^{at}P_n^*(t)/(1-\lambda)=hs(z_{n,t})+o(1)$, if $n=1 \pmod h$, and 0 otherwise. The method used is to express P_n^* as the Fourier transform of the corresponding characteristic function. For the case $a=0$, the author sharpens a previous estimate of the characteristic function [Sevast'yanov, Uspehi Mat. Nauk (N.S.) **6** (1951), no. 6(46), 47–99; MR **13**, 763]. For the case $a>0$, he uses the fact that, in the limit, the characteristic function, properly scaled, corresponds to a distribution with a continuous density function. It is stated that a result analogous to Theorem 3 holds for branching processes with a discrete time parameter, and is obtainable by a similar method.

T. E. Harris.

Meisling, Torben. Discrete-time queuing theory. Operations Res. **6** (1958), 96–105.

$M/G/1$ is studied with the modification that customers only enter or leave the system at equally-spaced discrete time points. The mean queue-length at the times customers leave, and the mean waiting-time are found.

D. V. Lindley (Cambridge, England).

Gani, J. Problems in the probability theory of storage systems. J. Roy. Statist. Soc. Ser. B. **19** (1957), 181–206; discussion 212–233.

A valuable review of work on storage problems. The following topics are discussed: provisioning, the problem of emptiness and the optimal inventory policy; stationary distributions for finite and infinite dams; numerical methods.

D. V. Lindley (Cambridge, England).

Kendall, David G. Some problems in theory of dams. J. Roy. Statist. Soc. Ser. B. **19** (1957), 207–212; discussion 212–233.

A partial solution is provided to the problem of inverting the Laplace transform obtained by Moran [Quart. J. Math. Oxford Ser. (2) **7** (1956), 130–137] for the volume of water held in an infinite dam when the input is additive of the gamma type. In the same problem, but without the restriction to a gamma distribution, the distribution of time until the dam is first empty, given the initial amount in it, is obtained.

In the full and interesting discussion of this paper and the one reviewed above, H. E. Daniels obtains the inverse referred to above, D. V. Lindley and F. Downton extend Moran's results and W. L. Smith obtains an asymptotic form for the equilibrium dam-content distribution. Connections with queueing theory are emphasized by F. G. Foster and C. B. Winsten, and M. S. Bartlett suggest an input model which is not additive. There are also contributions of a more practical nature including a mathematically stimulating one on the range of rainfall distributions by Hurst. The results of the discussion are summarized by Kendall in his reply.

D. V. Lindley (Cambridge, England).

Tortrat, Albert. Itération de certaines matrices et processus de Markoff. C. R. Acad. Sci. Paris **245** (1957), 1872–1874.

A novel method for computing the distribution of the one-dimensional (skew) random walk with one or two boundaries, using generating functions, Kelvin images, and contour integration. *H. P. McKean, Jr.*

Elldin, A. Brief presentation of the theory of telephone traffic. Ericsson Rev. **25** (1958), 13–22.

The paper gives a largely informal account of the usual mathematical models for telephone traffic, mentioning the work of Erlang and Palm. The definitions of loss and delay systems, of gradings, and of link systems are sketched, and problems of calculating loss and delay, of economic design, and of traffic measurement are touched upon. *V. E. Beneš* (Murray Hill, N.J.).

Blachman, Nelson M. On Fourier series for Gaussian noise. Information and Control **1** (1957), 56–63.

It is well known that for finite T, the constants C_m, in the Fourier series representation, $f(t)=\sum_{m=-\infty}^{\infty} C_m e^{2\pi mit/T}$, $-\frac{1}{2}T<t<\frac{1}{2}T$, of a stationary Gaussian process, $f(t)$, are not independent. The author gives a detailed exposition of this fact, and its ramifications. *E. Reich.*

Blachman, Nelson M. Limiting frequency-modulation spectra. Information and Control **1** (1957), 26–37.

The author considers a frequency modulated carrier $F(t)=2^{\frac{1}{2}}\cos[\Omega t+\phi(t)]$, $\phi(t)=\int_0^t m(t')dt'$, where the modulation signal $m(t)$ is a stationary stochastic process. His purpose is to obtain approximations to the autocorrelation function, $\Psi(\tau)$, of $F(t)$, and the power spectrum, $W(\omega)$, for the limiting cases of small and large values of $\langle[m(t)]^2\rangle_{\text{av}}$. The relation of the results to work of Woodward, Middleton, and others is discussed. *E. Reich.*

See also: Statistics: Leipnik; Akaike; Jenkins and Priestly; Lomnicki and Zaremba; Whittle; Rosenblatt; Hannan; Goodman; Campbell. **Relativity:** Chernikov. **Astronomy:** Ramakrishnan and Vasudevan. **Economics, Management Science:** van Klinken; Peck and Hazelwood. **Programming, Resource Allocation, Games:** Gillette. **Information and Communication Theory:** Khinchin.

STATISTICS

★**Kendall, Maurice G.; and Buckland, William R. A dictionary of statistical terms.** Published for The International Statistical Institute by Oliver & Boyd, London-Edinburgh; and Hafner Publishing Company, New York, 1957. x+493 pp. $4.50.

This dictionary provides an explanation of terms in current use, whether they are, in the authors' judgement, desirable or not; only a few elementary terms which are self-explanatory are omitted. Glossaries in French, German, Italian and Spanish are included. An impressive list of collaborators vouchsafes authoritative entries.

★**Roy, S. N. Some aspects of multivariate analysis.** John Wiley and Sons Inc., New York; Indian Statistical Institute, Calcutta, 1957. viii+214 pp. $8.00.

This monograph does not attempt to cover the entire area of multivariate analysis, the statistical analysis of samples from multivariate normal populations, or even a major part of it. The monograph deals with samples of fixed size, and the main emphasis is on obtaining confidence bounds on certain parametric functions that are a set of natural measures of deviation from a null hypothesis. The first twelve chapters lead up to the confidence bounds which are discussed in detail in chapters 13 and 14. In chapter 15, the last of the monograph, some nonparametric generalizations of analysis of variance and multivariate analysis applied to categorized data in contingency tables are considered [see Roy and Mitra, Biometrika **43** (1956), 361–376; MR **18**, 522].

The underlying basis is a heuristic class of tests and a union-intersection principle [see Roy, Ann. Math. Statist. **24** (1953), 220–238; **25** (1954), 752–761; MR **15**, 241; **16**, 382; Roy and Bose, ibid. **24** (1953), 513–536; MR **15**, 726].

Included in the first twelve chapters are discussions of the multivariate normal population and the properties of samples therefrom, distribution problems, least squares, analysis of variance and generalizations, and bounds on the power functions. About 37% of the monograph contains nine useful appendices covering results in matrix theory, quadratic forms, transformations, roots of determinantal equations, Jacobians, canonical reduction of certain distribution problems, and integration, in particular, that connected with the distributions of the smallest and largest roots of a determinental equation.

The multivariate confidence bounds discussed are those for multivariate normal populations relating to (a) the mean vector; (b) mean differences in k populations; (c) the general linear hypothesis; (d) multicollinearity of means; (e) the covariance matrix; (f) characteristic roots related to a pair of covariance matrices; (g) regression-like parameters.

Chi-square test criteria and results on large-sample distributions are given in relation to the contingency table hypotheses in chapter 15.

The format, typography, editing, exposition, and proof-reading of this monograph are all, unfortunately, poor, and tend to distract and annoy the reader. An index is not included. The expressions "well-known" and "it is easy to" are overused, and applied over a range from bare statement to results proved in detail. There is no point in belaboring this aspect with many specific instances.

Many further results and proofs are promised for later monographs. This monograph is admittedly not adequate for the needs of a possible user of statistics, but despite its defects will find a useful place amongst the books of the advanced student or research worker in mathematical statistics. *S. Kullback.*

★**Kendall, M. G. A course in multivariate analysis.** Griffin's Statistical Monographs & Courses, no. 2. Hafner Publishing Company, New York, 1957. 185 pp.

This monograph covers the material in a set of lectures as given by Professor Kendall in 1954 and 1955 at North Carolina, at Virginia Polytechnic Institute, and at London School of Economics, though it is revised to include material published during 1956 and 1957. The purpose appears to be a survey of important methods which are useful in multivariate analysis rather than a formal development of the basic theory. The author has used numerous

illustrations, frequently taken from previous literature, and has concerned himself with the numerical solutions of problems as well as the theoretical basis of the solution. His attitude is illustrated in his introductory chapter where he gives concreteness to the presentation by suggesting practical multivariate problems in such fields as biometrics, education, agriculture, sociology, medicine, physics, anthropology, economics, experimentation and industry. Certain remarks, and the use of certain materials, indicate that the author had the psychologist especially in mind, but the monograph might be studied profitably by anyone who wishes to survey available methods of multivariate analysis.

The presentation proper begins with chapter 2 and deals with component analysis in which components of the observations are sought. There is a discussion of principal components and an iterative solution of the characteristic equation. The centroid method of matrix factorization is described. The author then moves on, in Ch. 3, to a discussion of factor analysis in which a certain factor model is assumed and an attempt is made to fit the data to it. This leads to a discussion of the so-called "communalities" and to some methods which have been proposed by psychologists for analysis into factors. In general, the present sampling theory for these methods is very inadequate.

Ch. 4 deals with the subject of functional relationship. There is a discussion of the relationship between two variables when each is subject to error. Several alternatives are examined and the discussion appears somewhat more technical than in the earlier chapters. Ch. 5 is devoted to canonical analysis. The earlier sections apply canonical methods in the reduction of regression problems. The later sections deal with canonical correlation. There are several illustrations.

Ch. 6 deals with problems of sampling and includes brief discussions of the distribution of Wishart, the distributions of the correlation determinant, the distribution of Hotelling T^2, and the distribution of characteristic roots. There is a discussion of significance and estimation in component and factor analysis. Ch. 7 continues with notes on the history of multivariate analysis. Wilks' criterion is discussed and there are historical notes on latent roots and on discriminant analysis.

Ch. 8 discusses tests for homogeneity, including the Pearson-Wilks results for bivariate populations and the analysis of dispersion. Ch. 9 examines the question of discriminatory analysis and extends the discussion to the case of k populations. The monograph terminates with a list of over 100 references and a list of 17 exercises. There are some references in the text which are not included in the formal list of references.

This monograph should serve as an excellent introduction for one who wishes to know about the methods of multivariate analysis. It can also be used by the student of theory as a means for locating important contributions of the last 30 years. *P. S. Dwyer.*

Brown, R. L.; and Fereday, F. Multivariate linear structural relations. Biometrika **45** (1958), 136–153.

Let $x_1, x_2, \cdots, x_n$ be n observations of a vector variate X. This variate is normally distributed with the unit matrix as variance covariance matrix. The following linear relationship exists: $\alpha_0+\alpha'X=0$ (α_0 a scalar, α a vector). The statistic $n\phi=\sum_{j=1}^{n}(\alpha_0+\alpha'x_j)'(\alpha_0+\alpha'x_j)/(\alpha'\alpha)$ has a χ^2 distribution with n degrees of freedom. Hence we can find a value ϕ_p for a given probability level p.

Let U be the variance-covariance matrix of the observations. $\phi=(\alpha_0{}^2+\alpha'U\alpha)/\alpha'\alpha$. Let Λ be a diagonal matrix of the eigenvalues of U, and L a matrix of the eigenvectors. The envelope of all linear relations may be written, in terms of the transformed variables $Y=L'X$, $1+Y'(\Lambda-\phi I)^{-1}Y$. [R. L. Brown, Biometrika, **44** (1957), 84–96; MR **19**, 186]. A confidence quadric is computed and the best relationship established. The whole approach is related to the test for multicollinearity proposed by Tintner [Ann. Math. Statist. **16** (1945), 304–308; MR **7**, 132], but this method is based upon a "perpendicular" definition of errors. The two methods are compared and it is shown that, at least in a special case, the present method is more selective. There are 3 artificially constructed examples, which show that both methods yield the same conclusions. *G. Tintner* (Ames, Iowa).

Gold, Louis. Generalized Poisson distributions. Ann. Inst. Statist. Math., Tokyo **9** (1957), 43–47.

The author calls $cq^xe^{-qu}\sum_{j=x}^{\infty}u^j/j!$ with c=const and $0<q<\infty$ the generalized Poisson (G.P.) distribution, since it is a generalization of the Poisson exponential binomial limit (PEBL) $ce^{-u}\sum_{j=x}^{\infty}u^j/j!$ when $q\to 1$, which in turn approaches the simple Poisson (S.P.) when $u\to 0$. Expressions are obtained for the moments of the G.P. which arises in connection with the molecular size of a certain class of polymerization reactions and also for the PEBL which appears in telephone trunking and queuing theory. *H. P. Edmundson* (Santa Monica, Calif).

Hudimoto, Hirosi. A note on the probability of the correct classification when the distributions are not specified. Ann. Inst. Statist. Math., Tokyo **9** (1957), 31–36.

The author proves some modifications of certain of his earlier results [same Ann. **8** (1956), 105–112; MR **19**, 472] for instance: If $F_i(x)$ are continuous and if α and η satisfy the condition $P(|N-p|<\eta)\geqq 1-\alpha$, then we have

$$P\Big[\widehat{c(x)}-\Big\{\eta+p\Big(\frac{1}{2N(p-\eta)}\log\frac{1}{\alpha_1}\Big)^{\frac{1}{2}}+q\Big(\frac{1}{2N(q-\eta)}\log\frac{1}{\alpha_2}\Big)^{\frac{1}{2}}\Big\}\leqq c(x)\Big]\geqq(1-\alpha)(1-\alpha_1-\alpha_2)$$

for any given positive numbers α_i with $0.001\leqq\alpha_i\leqq 1$ ($i=1, 2$). [For the notations see the above cited review]. For the proof, he uses a result of Z. W. Birnbaum and F. H. Tingey [Ann. Math. Statist. **22** (1951), 592–596; MR **13**, 367]. *L. Schmetterer* (Hamburg).

Cox, D. R. Some statistical methods connected with series of events. J. Roy. Statist. Soc. Ser. B. **17** (1955), 129–157; discussion, 157–164.

Bartholomew, D. J. Tests for randomness in a series of events when the alternative is a trend. J. Roy. Statist. Soc. Ser. B. **18** (1956), 234–239.

Let T be a random variable with probability density $p(T)$, such that $p(T)=0$ for $T<0$ and $T>1$, and let $0\leqq T_1\leqq T_2\leqq\cdots\leqq T_n\leqq 1$ be an ordered sample of T. The statistic $m=(\sum_{i=1}^{n}T_i)/n$ has been used to test the hypothesis $p(T)=1$, $0\leqq T\leqq 1$. The author considers the alternatives $p(T;\beta)=e^{\beta T}\beta/(e^{\beta}-1)$, $0\leqq T\leqq 1$, introduced by D. R. Cox in the paper listed above. For these alternatives he has: (1) an explicit expression for the power of the test based on m, which lends itself to computation; (2) an iterative procedure for computing the power of the test based on Kolmogorov's statistic D_n; (3) an explicit ex-

pression for the power of the test based on the one-sided statistic D_n^+. Numerical computations for $n=5$ indicate that the m-test is somewhat more powerful than the D_n and the D_n^+ tests. *Z. W. Birnbaum.*

Leipnik, Roy B. Moment generating functions of quadratic forms in serially correlated normal variables. Biometrika **45** (1958), 198–210.

The author uses a method of Kac [Trans. Amer. Math. Soc. **59** (1946), 401–414; MR **8**, 37] and Kac and Erdös [Bull. Amer. Math. Soc. **52** (1946), 292–302; MR **7**, 459] to obtain a closed form for the joint generating function of X_1^2, $\sum_2^n X_j^2$, X_{n+1}^2, $\sum_1^n X_jX_{j+1}$, where the X_j's are serially correlated normal variables. Exact moments of several estimates of the variance and autocorrelation are computed in the case of the Ornstein-Uhlenbeck process. The estimates are compared in terms of the asymptotic behavior of their moments. The function $A(t)$ entering into a formula given by the author for the covariance function of a serially stationary process cannot be arbitrary if the formula is to yield a true covariance function. *M. Rosenblatt* (Bloomington, Ind.).

Seal, K. C. Approximate distribution of certain linear function of order statistics. Sankhyā **17** (1957), 345–348.

The author is concerned with the approximate normality of a convex combination of the order statistics from a $N(0, 1)$ distribution function. An empirical study for samples of size four is discussed. *I. Olkin.*

Corsten, L. C. A. Partition of experimental vectors connected with multinomial distributions. Biometrics **13** (1957), 451–484.

An expository account of the orthogonal partition of contingency — and in particular the 2^2 and 2^3 — tables for χ^2 analysis leads to a discussion of the interpretation of the various interaction terms. *H. L. Seal.*

Lord, Frederic M. A significance test for the hypothesis that two variables measure the same trait except for errors of measurement. Psychometrika **22** (1957), 207–220.

Let x_1, x_2, x_3, x_4 be jointly normal, with a covariance matrix (σ_{ij}) restricted by the conditions $\sigma_{11}=\sigma_{22}$, $\sigma_{33}=\sigma_{44}$, $\sigma_{13}=\sigma_{14}=\sigma_{23}=\sigma_{24}=\sigma$ (say), and $\sigma^2\leq\sigma_{12}\sigma_{34}$. The maximum likelihood estimates of the σ's are derived, and the likelihood ratio test is developed for the additional hypothesis that $\sigma^2=\sigma_{12}\sigma_{34}$. The problem arises in the situation indicated in the title. *G. Elfving.*

Graybill, Franklin A.; and Marsaglia, George. Idempotent matrices and quadratic forms in the general linear hypothesis. Ann. Math. Statist. **28** (1957), 678–686.

The applicability of Cochran-like theorems in the analysis of variance is extended by means of some theorems about idempotent matrices. These theorems are easy extensions of elementary facts about projection operators in a linear vector space. In the reviewers' opinion, the linear algebra technique which is current in some expositions of the analysis of variance is a much more direct approach to this subject and requires much less computation. *M. Dwass* (Evanston, Ill.).

Thionet, Pierre. Sur la variance de l'estimation d'une variance. C. R. Acad. Sci. Paris **245** (1957), 2168–2170.

A simple method for calculating the variance of the sample estimate of the variance of a finite population is described. *D. V. Lindley* (Cambridge, England).

Thionet, Pierre. Une généralisation de la variance d'échantillonnage dans le cas de tirages exhaustifs d'une urne. C. R. Acad. Sci. Paris **245** (1957), 2464–2467.

Let $p(N, n)$ be a positive function of N, the size of a finite population, and n, the size of a sample therefrom. If

$$p(N, n')=p(N, n)+Ep(n, n')$$

whenever $n'<n$, then $p(N, n)$ is called a loss of information incurred in knowing only the sample and not the whole population. (The expectation is taken over all samples of size n from which a sub-sample of size n' could be taken.) It is shewn that the variances of the sample means and variances are such functions. Other remarks concern graphical representation and 'distance' ideas. *D. V. Lindley* (Cambridge, England).

Thionet, Pierre. Représentation topologique des sondages. C. R. Acad. Sci. Paris **246** (1958), 46–49.

Extensions of the ideas of the previous paper to sampling in several stages. It is shewn that the variance of an unbiased estimate is a loss of information in the sense of the previous paper. *D. V. Lindley.*

Creasy, Monica A. Analysis of variance as an alternative to factor analysis. J. Roy. Statist. Soc. Ser. B. **19** (1957), 318–325.

Suppose two tests are taken by a group of p people, yielding scores x_{ij} ($i=1, 2$; $j=1, 2, \cdots, p$), and that the scores are normally distributed and standardized: $\sum_j x_{ij}=0$, $\sum_j x_{ij}^2=p-1$. Then it is suggested that an analysis of variance of the scores and a test of the mean square for people against the mean square for interaction would be suitable for testing the presence of a general factor, and would be approximately valid for $p>40$. The extension to n tests is immediate. If n_1 tests contain a factor A, apart from the general factor, and the remaining $n-n_1$ do not, a similar analysis with people tested against the interaction of people with A (each on $p-1$ degrees of freedom) provides a test for the factor. The interactions of several factors are discussed and numerical illustrations included. *D. V. Lindley* (Cambridge, England).

Smith, H. Fairfield. A multivariate analysis of covariance. Biometrics **14** (1958), 107–127.

This is a re-examination of an analysis of covariance with several characters previously discussed by another author. The point is first made that a standard routine analysis of variance reduction is inappropriate when the "independent" variable is affected by treatments.

The treatments are combinations of time and drugs. An analysis is proposed which obtains regressions on time for each drug and leads to the selection of a convenient function of time observations in which available information about contracts between drugs is concentrated. Responses of dependent variates to drugs are then studied by means of a structural analysis in which random errors of both variates are considered. *P. S. Dwyer.*

Steinhaus, H. The problem of estimation. Ann. Math. Statist. **28** (1957), 633–648.

This expository paper demands little knowledge of statistical theory other than an understanding of expected value. Minimax point estimation for the binomial, multinomial and Poisson are treated with various forms of squared error loss function. *L. E. Moses.*

Voelz, K. Über die Berechnung der Regressionskoeffizienten mit Hilfe von Orthogonalfunktionen. Mitteilungsbl. Math. Statist. **9** (1957), 113–129.
Expository paper. *P. Meier* (Baltimore, Md.).

Akaike, Hirotugu. On optimum character of von Neumann's Monte Carlo model. Ann. Inst. Statist. Math., Tokyo **7** (1956), 183–193.

A Markov process has an absorbing state k and a finite number of other states, absorbing or not. The transition probabilities are known. It is desired to estimate by random sampling the quantity T, the probability that a particle eventually goes to the state k. The author discusses a splitting technique in which a particle is transformed into a random number of particles each time it changes state, a variable multiplicative factor being applied to each particle born to compensate for the splitting. The estimate of T is the total weight of all particles that reach k, and it is desired to study sampling schemes that reduce the variance of the estimate. On the basis of a study of variances, a scheme is recommended that is similar to one proposed for neutron shielding problems by v. Neumann [for discussion of the latter scheme see H. Kahn and T. Harris, Nat. Bur. Standards, Appl. Math. Ser. no. **12** (1951), 27–30; MR **13**, 162]. The author's recommended scheme has the additional practical advantage, as does von Neumann's, that all particles arriving at k have the same weight. *T. E. Harris.*

Akaike, Hirotugu. Monte Carlo method applied to the solution of simultaneous linear equations. Ann. Inst. Statist. Math., Tokyo **7** (1956), 107–113.

The author proposes a matrix inversion technique similar to that described by Forsythe and Leibler [Math. Tables Aids Comput. **4** (1950), 127–129; MR **12**, 361], based on ideas of Ulam and v. Neumann, except that at each step in the sampling process, there may be splitting. (See the preceding review.) The author finds the variance of his estimate; it is always finite. It is stated that experimental results will be presented later. *T. E. Harris.*

Bliss, C. I.; and Owen, A. R. G. Negative binomial distributions with a common k. Biometrika **45** (1958), 37–58.

Estimates for the common dispersion parameter of a negative binomial distribution, when several sets of counts are available, are described and illustrated with numerical examples. *I. Olkin* (Stanford, Calif.).

Kloos, J.; and Turner, F. The determination of a factor of safety on the basis of a single probability parameter. Svenska Aeroplan A. B. Tech. Note No. **37** (1957), 13 pp.

Given a random sample from a normal distribution with unknown mean, a statistic is to be found such that, prior to sampling, the expected proportion of the distribution above this statistic equals a given value. The cases of known variance, unknown variance, and known coefficient of variation are treated. {Although the authors seem to use some type of fiducial arguments, all three cases can be treated by ordinary probability arguments, using in the first two cases the same statistics as in the present paper, and in the third case the statistic $a\bar{x}$, where $\bar{x}$ is the sample mean and a a constant to be determined.} *D. M. Sandelius* (Göteborg).

Bahadur, R. R. On unbiased estimates of uniformly minimum variance. Sankhyā **18** (1957), 211–224.

It was shown by Lehmann and Scheffé [Sankhyā **10** (1950), 305–340; MR **12**, 511] that if in a statistical situation there exists a complete sufficient statistic, then every estimable function possesses an unbiased estimate with uniformly minimum variance. The author proves the converse of this theorem. He also proves that quite generally there exists a statistic such that the totality of bounded estimates with uniformly minimum variance coincides with the totality of bounded functions of this statistic. The proofs require only mild assumptions, essentially that the sample space is Euclidean and that the family of distributions is dominated. A basic tool for the proofs is a theorem of Stein [Ann. Math. Statist. **21** (1950), 406–415; MR **12**, 192] concerning the existence and uniqueness of locally efficient estimates. The author gives a simple proof of this theorem and applies it to some problems regarding lower bounds to the variance of unbiased estimates. *E. L. Lehmann* (Berkeley, Calif.).

Roy, Jogabrata; and Mitra, Sujit Kumar. Unbiassed minimum variance estimation in a class of discrete distributions. Sankhyā **18** (1957), 371–378.

In this paper, for a wide class of discrete distributions involving one unknown parameter, the uniformly minimum variance unbiased (UMVU) estimate of the parameter is derived and the UMVU estimate of its variance is also obtained. Tables of the UMVU estimate are given for the Poisson distribution truncated at zero, for sample size up to 10. (From the author's summary.) *E. L. Lehmann.*

Armitage, P. Numerical studies in the sequential estimation of a binomial parameter. Biometrika **45** (1958), 1–15.

The author describes a method for obtaining confidence limits for the parameter in a binomial distribution when the sampling is sequential. This method is applied to three closed sequential stopping rules and the results numerically compared with confidence limits given by the fixed sample size formulae. Tables of numerical results are included. The distributions of the fixed sample size maximum likelihood estimator and an unbiased estimator of the binomial parameter are presented for the examples considered. *F. C. Andrews* (Eugene, Ore.).

Linnik, Yu. V. Some remarks on least squares in connection with direct and inverse location problems. Teor. Veroyatnost. i Primenen. **2** (1957), 349–359. (Russian. English summary)

The author considers the estimation of an n-dimensional column-vector A satisfying the relation $Y=X^{(0)}+XA$, where X is a known $N\times n$ matrix, $X^{(0)}$ is known, and the result of observation is $L=Y+\Delta$, the vectors $X^{(0)}$, Y, L, and Δ being N-dimensional. He supposes that the error Δ is a normal unbiased vector with correlation-matrix σ^2P^{-1}, where P is a known diagonal matrix whose diagonal elements (the weights) are positive. The author establishes the following theorem. Let G be a known $m\times n$ matrix with rank m, let $H=GA$, let Z be the general (m-dimensional) coordinate-vector, let $C=X^TPX$ and $K=GC^{-1}G^T$, let $\bar{A}$ be the least-squares estimate of A, and let $\bar{V}=X^{(0)}+X\bar{A}-L$ and $\bar{H}=G\bar{A}$. Then K is nonsingular and the confidence-ellipsoid

$$(Z-\bar{H})^T\cdot K^{-1}\cdot(Z-\bar{H})=\gamma_0(\bar{V}^TP\bar{V})$$

encloses the point $Z=H$ with probability $p_0=F_{m,N-n}(\gamma_0)$, where $F_{m,N-n}(x)$ is Fisher's cumulative distribution function (for the ratio of two χ^2-variates with m and $N-n$ degrees of freedom). With appropriate special

choices of G we get the well known separate confidence intervals for the individual components of A [cf. A. N. Kolmogorov, Uspehi Mat. Nauk (N.S.) **1** (1946), no. 1(11), 57–70; MR **8**, 523]. The author applies his theorem to certain location problems in geodesy. In these cases $n=2$, and the equation $F_{2,N-2}(y_0)=p_0$ can be solved in terms of elementary functions. *H. P. Mulholland* (Exeter).

Robson, D. S. Applications of multivariate polykays to the theory of unbiased ratio-type estimation. J. Amer. Statist. Assoc. **52** (1957), 511–522.

The author considers unbiased ratio estimators of the type

$$y'=r\bar{X}+\left(\frac{N-1}{N}\right)\left(\frac{n}{n-1}\right)(\bar{y}-r\bar{x}),$$

where $\bar{y}$, $\bar{x}$ and r are respectively arithmatic means of values of y_i, x_i and r_i attached to a random sample of size n, drawn from a population of size N. The quantities r_i are the ratios of y_i/x_i. This estimator was introduced by Hartley and Ross (1954) as an unbiased estimator of the population mean $\bar{Y}$ and depends on the knowledge of the population mean $\bar{X}$. Whilst the variance of this estimator was given by Goodman and Hartley (1956) only for the case of very large N, the author here is able to derive the exact variance formula for the case of finite N. He is able to do this by a novel application of multivariate polykays which are akin to those introduced by Tukey (1956). The resulting formula, although of considerable complexity, is nevertheless required for situations when the sampling fraction is large. *H. O. Hartley* (Ames, Iowa).

Fabian, Václav; and Špaček, Antonín. Correction to "Experience in statistical decision problems". Czechoslovak Math. J. **6(81)** (1956), 434.

The title article is in same J. **6(81)** (1956), 190–194 [MR 19, 189].

Kiefer, J. Invariance, minimax sequential estimation, and continuous time processes. Ann. Math. Statist. **28** (1957), 573–601.

The author's main purpose is to prove, by the method of invariance, that in a wide class of sequential decision problems there exists a minimax procedure δ^* among the class of all sequential decision functions such that δ^* observes the process for a constant length of time. The precise assumptions for the validity of this result are, as customary in decision theory, much too involved to be reproduced here. The main tool is a general invariance theorem [generalizing one due to M. P. Peisakoff, Thesis, Princeton, 1951], which is proved under conditions easily verifiable in important cases. From the general theorem many known and new special results are derived. The paper abounds in examples and counter-examples. It contains a critical evaluation of the assumptions and a lucid discussion of the various methods for proving the minimax character of decision functions. *A. Dvoretzky*.

Kiefer, J. Optimum sequential search and approximation methods under minimum regularity assumptions. J. Soc. Indust. Appl. Math. **5** (1957), 105–136.

This is the first paper of any length devoted to a detailed discussion of the interesting and difficult mathematical theory of sequential search processes. Problems of this nature arise not only in the theory of testing, but also in the field of computing. The results are stated in the language of sequential analysis and decision theory, with a number of explicit solutions given for various one-dimensional search processes. Multi-dimensional results are sparse. The connection between these problems and the Monro-Robbins stochastic approximation method is indicated [Ann. Math. Statist. **22** (1951), 400–407; MR **13**, 144]. Further unpublished results of Johnson, and Johnson and Gross concerning the location of the maximum and zero of a convex function may be found in R. Bellman, "Dynamic programming" [Princeton, 1957; MR **19**, 820]. *R. Bellman*.

Pompilj, Giuseppe. Lo schema delle prove ripetute a due stadi stratificati. Fac. Sci. Statist. Demogr. Attuar. Ist. Statist. Ist. Calcolo Probab. Publ. no. **3** (1956), 22 pp.

This paper gives a detailed computation of the variance of the natural unbiased estimator of the population average in a special multistage stratified cluster sampling plan. The finite population is divided into strata, and from each a sample of primary clusters is chosen. Each primary cluster is made up of secondary clusters, and separate samples of individuals from all secondary clusters of the chosen primary clusters are taken. Otherwise, sampling rates are fixed but arbitrary. All sampling is random, without replacement. The variance of the natural estimator of the population average is computed by using indicator random variables (1 or 0 depending on whether some element does or does not appear in the sampling).

The result of the paper is fairly standard, and may be obtained, for example, as a special case of (12.3), Sec. 12, Chap. 7 of "Sample survey methods and theory", vol. I [Wiley, New York, 1953; MR **15**, 332] by M. H. Hansen, W. N. Hurwitz and W. G. Madow. The paper gives no references to the extensive recent sampling literature outside of Italy. *W. Kruskal* (Chicago, Ill.).

Olkin, Ingram. Multivariate ratio estimation for finite populations. Biometrika **45** (1958), 154–165.

In sample surveys precision in estimating the unknown mean $\bar{Y}$ of a finite population may be increased by using auxiliary variables which are correlated with Y. Consider the following model for the population: $Y_1, \cdots, Y_N$, $\bar{Y}$ unknown, $X_{11}, \cdots, X_{1N}$, $\bar{X}_1 \neq 0$ known, $R_1=\bar{Y}/\bar{X}_1, \cdots, X_{p1}, \cdots, X_{pN}$, $X_p \neq 0$ known, $R_p=\bar{Y}/\bar{X}_p$, and the known $(p+1)\times(p+1)$ covariance matrix S. A simple random sample $(y_i, x_{1i}, \cdots, x_{pi})$ $(i=1, \cdots, n)$, from the population is observed. The proposed ratio estimate of $\bar{Y}$ is $\bar{y}'=w_1r_1\bar{X}_1+\cdots+w_pr_p\bar{X}_p$, where $r_i=\bar{y}/\bar{x}_i$ and $w=(w_1, \cdots, w_p)$, $\sum_i w_i=1$ determined to minimize the variance. An estimate of $V(\bar{y}')$ is given and comparisons made between mean estimation using simple random sampling and ratio estimation and between univariate and multivariate ratio estimation. An example is discussed where the population consists of the number of inhabitants in 200 large cities in 1930 and a sample of size 50 is taken, with $\bar{Y}$, $\bar{X}_1$ and $\bar{X}_2$ the 1950, 1940 and 1930 mean number of inhabitants. *S. Kullback* (Washington, D.C.).

Jenkins, G. M.; and Priestley, M. B. The spectral analysis of time-series. J. Roy. Statist. Soc. Ser. B. **19** (1957), 1–12 (discussion 47–63).

This is a survey of methods suggested for estimating the spectrum of a stationary Gaussian process. The authors describe and discuss some of these methods and also the problem of determining confidence intervals for the density and distribution functions of the spectrum. They

emphasize the difficulties which arise for mixed spectra having both continuous and discrete components. A description is given of the programmes available for the electronic computer DEUCE for computing correlations and spectra. *U. Grenander* (Providence, R.I.).

Lomnicki, Z. A.; and Zaremba, S. K. On estimating the spectral density function of a stochastic process. J. Roy. Statist. Soc. Ser. B. **19** (1957), 13–37 (discussion 47–63).

This paper discusses optimality questions related to the estimation of the spectral density of a linear stationary process. As an optimality criterion, the authors use the expected integrated mean square criterion introduced by the reviewer [Ark. Mat. **1** (1952), 503–531, p. 521; MR **14**, 187]. The best estimate is defined as the one minimizing this criterion. It should be noted that one must know the spectrum in order to be able to construct the best estimate of it; it is actually a definition of local optimality.

The authors discuss the relation between the resolvability and reliability of an estimate. They claim that these two quantities are not antagonistic. This is discussed in more detail in the Discussion on the Symposium, pp. 47–63; the consensus seems to be, however, that band width and variance oppose each other, but opinions may vary on the relation between resolvability of an estimate and the corresponding bandwidth of its spectral window.

A number of special estimates are discussed and compared, and the results of the paper are applied to numerical data. *U. Grenander* (Providence, R.I.).

Whittle, P. Curve and periodogram smoothing. J. Roy. Statist. Soc. Ser. B. **19** (1957), 38–47 (discussion 47–63).

The author starts from the observation that "the only justification for smoothing an empirical curve at all is the knowledge that the parent curve being estimated is itself smooth". Concluding that the best smoothing formula is the one that best utilizes the information on the smoothness properties of the curve, the author is led to introduce a priori distributions in the space of all the parent curves.

This is carried out in the case of estimating the spectral density of a stationary Gaussian process. Assuming a particular functional form for the population moments, an expression is derived for the optimal estimate. This is compared to estimates that have been proposed earlier.

This procedure is also applicable when the curve to be estimate dis a probability density, and this possibility is discussed briefly. *U. Grenander* (Provicence, R.I.).

Rosenblatt, Murray. A multi-dimensional prediction problem. Ark. Mat. **3** (1958), 407–424.

This paper deals with the linear prediction of multidimensional stationary stochastic processes, which has recently attracted a good deal of attention. The covariance of the errors associated with the best linear predictor has not been given in a really explicit form so far, but it is known that the determinant can be expressed simply in terms of the spectrum of the process. The author investigates when exact prediction is possible and gives conditions for this. He points out that the real source of the difficulties that do not appear in the scalar case is that the logarithmic function of a matrix does not satisfy the equation $\log MN=\log M+\log N$ unless the two matrices commute. There are two special cases where this occurs: when the cospectrum is zero or when the quadrature spectrum is zero. For these two cases the author determines the covariance matrix of the error in a one-step prediction. The prediction problem is also discussed in the context of W. J. Pierson's theory of storm-generated ocean waves. *U. Grenander* (Providence, R.I.).

Hannan, E. J. The variance of the mean of a stationary process. J. Roy. Statist. Soc. Ser. B. **19** (1957), 282–285.

The author discusses a statistic proposed by Jowett [same J. **17** (1955), 208–227; MR **17**, 869] to estimate the variance of the mean of n consecutive observations on a stationary process. Modifying Jowett's estimate, the author obtains another one, which is related to the smoothed periodogram. It is pointed out that the variance in question is approximately proportional to the spectral density at zero frequency and hence that the problem is related to the estimation of a spectral density. This leads to estimates of more general form. The various estimates are computed and discussed for a sample of $n=60$ observations from a simple Markov process.

U. Grenander (Providence, R.I.).

★ **Goodman, N. R. On the joint estimation of the spectra, cospectrum and quadrature spectrum of a two-dimensional stationary Gaussian process.** Engineering Statistics Laboratory, College of Engineering, New York University, New York, Scientific paper no. 10, Nobs-72018 (1734–F) and Nonr–285 (17), (1957). v+168 pp.

This paper contains the first extensive discussion of statistical spectral analysis of stationary time series in a multidimensional context. A real-valued two-dimensional stationary Gaussian process is the basic model. The spectral distribution function of the process is assumed to be absolutely continuous with a continuous spectral density function (2×2 matrix-valued). A class of estimates of the spectral density functions of the two components of the process and the real and imaginary parts of the cross-spectral density function of the two components (the co and quadrature spectrum) are considered jointly. The estimates are linear forms in sample covariances. Asymptotic expressions for the covariance matrix of these four estimates are obtained. This extends work of Grenander and the reviewer [Proc. 3rd Berkeley Symposium on Math. Statist. and Probability, 1954–1955, vol. 1, Univ. of California Press, 1956, pp. 77–93; MR **18**, 946]. A distribution that the author calls the complex Wishart distribution is proposed on heuristic grounds as an approximation to the joint distribution of the spectral estimates. This distribution is the analogue in the multidimensional context of the χ^2 distribution proposed by Tukey [Sampling theory of power spectrum estimates, ONR Publ. Navexos **P–735**] as an approximation for the distribution of spectral estimates in the one-dimensional case. The results are used to obtain an estimate of the frequency response function of a linear time invariant system. *M. Rosenblatt.*

Campbell, L. Lorne. On the use of Hermite expansions in noise problems. J. Soc. Indust. Appl. Math. **5** (1957), 244–249.

Let $V(t)=S(t)+N(t)$ be the input of a detector, with $S(t)$ a deterministic signal and $N(t)$ stationary Gaussian noise. The detector is assumed to have no memory, that is, the output $I(t)=g(V(t))$. For a variety of detectors of this form, expressions are obtained for the autocorrelation function of the output (this may be an ensemble average, a time average or both) using Hermite expansions. The domain in which the results hold is at times not clearly indicated. $S(t)$ is implicitly assumed to be almost periodic.

The technique is an elaboration of that used by Thomson [Proc. Inst. Elec. Engrs. C. **102** (1955), 46–48; MR **16**, 1036] and Grenander and Rosenblatt [Statistical analysis of stationary time series, Wiley, New York, 1957; MR **18**, 959]. *M. Rosenblatt* (Bloomington, Ind.).

Craddock, J. M. An analysis of the slower temperature variations at Kew Observatory by means of mutually exclusive band pass filters. J. Roy. Statist. Soc. Ser. A. **120** (1957), 387–397.

The author decomposes some meteorological series into components of approximately distinct frequency ranges by applying appropriate moving averages (although the method of calculating the moving average coefficients is left unexplained). He proposes that the uniformity of the power spectrum be tested by treating the ratio of the variances of a pair of components as proportional to an F statistic. *P. Whittle* (Wellington).

Charnock, H. Notes on the specification of atmospheric turbulence. J. Roy. Statist. Soc. Ser. A. **120** (1957), 398–408.

The author is interested in separating the "mean" and "turbulent" motions of the atmosphere, and in obtaining relations between the statistics of the two components. He describes some techniques used by meteorologists, related to a crude form of covariance analysis, and then describes an analysis into frequency ranges by the use of band-pass filters. *P. Whittle* (Wellington).

Rushton, S.; and Neumann, J. Some applications of time series analysis to atmospheric turbulence and oceanography. J. Roy. Statist. Soc. Ser. A. **120** (1957), 409–439.

A valuable survey paper. The authors review some of the ideas of spectral analysis, give an account of some statistical analyses that have been made of meteorological and oceanographical data, and make several sensible remarks, e.g., the last one.

The discussion of the above three papers includes the suggestion by M. B. Priestley of a method for dealing with mixed spectra. *P. Whittle* (Wellington).

Chow, C. K. An optimum character recognition system using decision functions. I. R. E. Trans. **EC-6** (1957), 247–254.

The only mathematical novelty is an example using the Neyman-Pearson Lemma to obtain a restricted-Bayes solution. The restriction (unorthodox in that it depends on the same prior distribution) is to decision functions with specified component of expected loss arising from a given subset of decisions. *J. Hannan.*

See also: Probability: Konijn; Goodman; Davenport and Root. **Quantum Mechanics:** Barut.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

★**Finzi, Bruno; e Udeschini, Paolo. Esercizi di meccanica razionale. 3a edizione ampliata.** Libreria Editrice Politecnica Cesare Tamburini, Milano, 1958. viii+546 pp.

This is an attractive collection of problems on rational mechanics and a good supplement to Finzi's textbook [Meccanica razionale, 2 vols., Zanichelli, Bologna, 1950; MR **14**, 325]. The authors stress the way of thinking which a student ought to follow: posing of the physical question; translation into mathematical language; solution of the equations; interpretation of the result. Hundreds of problems are arranged systematically; within each chapter they are given in an order of increasing difficulty. The problems are short, clearly drawn up and not too artifical. After each question some indications for the solution are given and the answer is added. The book ends with a recapitulation, containing mixed problems. The collection seems very valuable for the teaching of mechanics on a rather high level. The authors could be asked to give a separate chapter on small oscillations in a future edition. *O. Bottema* (Delft).

Lebowitz, Joel L. Modified virial theorem for total momentum fluctuations. Phys. Rev. (2) **109** (1958), 1464–1465.

The conventional virial theorem of Clausius relates the mean kinetic energy of a system of particles to the pressure and a "virial" of the interparticle force. In the present paper a similar virial theorem is given, which relates the fluctuations in the total linear momentum to a "virial" of the external forces acting on the system. The relationship is proven both classically and quantum mechanically. *M. J. Moravcsik* (Livermore, Calif.).

Blaschke, Wilhelm. Anwendung dualer Quaternionen auf Kinematik. Ann. Acad. Sci. Fenn. Ser. A. I. no. **250/3** (1958), 13 pp.

Die Drehungen des euklidischen Raumes um einen festen Punkt können in klassischer Weise mittels Quaternionen geschrieben werden: $x_1=Q'x_2Q$, wo x der Punktvektor und Q, Q' genormt und konjugiert sind. Verf. gibt eine sehr elegante Erweiterung auf die allgemeine Bewegung des Raumes; er wählt dabei statt des Punktes die gerichtete Gerade l als Element. Ist r ein Einheitsvektor auf l, x ein Punkt auf l, dann ist l durch r und $r_1=x\times r$ bestimmt (wobei $(rr)=1$ und $(rr_1)=0$) und also auch durch den dualen Vektor $\mathbf{r}=r+\varepsilon r_1$ mit $\varepsilon^2=0$. Es zeigt sich nun dass durch $\mathbf{r}_1=\mathbf{Q}'\mathbf{r}_2\mathbf{Q}$ die Bewegungen des Raumes dargestellt werden; dabei sind $\mathbf{Q}$, $\mathbf{Q}'$ genormte konjugierte duale Quaternionen. Man hat $\mathbf{Q}=\cos\omega+\mathbf{a}\sin\omega$, wo $\omega=\omega+\varepsilon\omega_1$, $\mathbf{a}=a+\varepsilon a_1$, $(\mathbf{aa})=1$; dabei ist a die Schraubenachse der Bewegung $\mathbf{Q}$, 2ω ihr Drehwinkel, $2\omega_1$ ihre Schiebung in Richtung a; $f(u+\varepsilon v)$ ist als $f(u)+\varepsilon vf'(u)$ definiert. Die Formeln werden auf die Kinematik zwangsläufiger Bewegungsvorgänge angewandt und dann auch auf die Bewegung von Punkten und Ebenen übertragen. Eine ausführlichere Darstellung wird angezeigt.

O. Bottema (Delft).

Schatz, Heinrich. Over the composition of motions in the space. Bull. Coll. Arts Sci. Baghdad **2** (1957), 84–91.

Vodička, Václav. Motion of a particle on an ellipsoid of revolution. Arch. Mech. Stos. **10** (1958), 107–114. (Polish and Russian summaries)

A solution, by elementary means, of the problem of the motion of a heavy particle on a vertical ellipse.

P. Franklin (Cambridge, Mass.).

Rektorys, Karel. Application of Lagrange's equations of the second category to the study of the function of a mechanism. Apl. Mat. **1** (1956), 319–333. (Czech. Russian and English summaries)

In the present paper the author studies the motion of a stone on the surface of a rotating cone. The problem is reduced to the approximate solution of Lagrange's equation of motion of the second kind. In particular, the author studies the influence of friction. The analysis is elementary.

K. Bhagwandin (Oslo).

Nowakowski, Władysław. Approximate spin analysis of a glider. Tech. Lotn. **12** (misprinted **13**) (1957), 71–75. (Polish)

The author considers a glider in an established steady spin. Using a number of simplifying assumptions he writes down the system of dynamical equations controlling this phenomenon. An approximate solution leads to a transcendental equation where the unknown is the angle of attack. The author solves this last equation using a primitive graphical method. He gives a numerical example of a glider Jaskolka-Bis, and obtains 38° for the angle of attack. He states that this is less than the critical angle of 45° without explaining why 45° is the critical angle. The validity of his assumptions and of his estimates of various constants is difficult to assess, but he claims that there is some correlation between his results and the experiment.

T. Leser (Aberdeen, Md.).

Karagodin, V. M. On the kinetic energy of a body of variable mass. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **63** (1956), 4–9. (Russian)

Karagodin, V. M. On a theorem concerning the kinetic energy of a body of variable mass. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **63** (1956), 9–12. (Russian)

Karagodin, V. M. On a generalization of Euler's equations. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **63** (1956), 12–22. (Russian)

Karagodin, V. M. Equations of motion of a body of variable mass in a moving system of coordinate axes. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **63** (1956), 22–27. (Russian)

Karagodin, V. M. Equations of motion of a body of variable mass in the natural system of coordinate axes. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **63** (1956), 27–31. (Russian)

Beletsky, V. V. Integrability of equations of motion of a solid around a fixed point under the action of a central Newtonian field of force. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 287–290. (Russian)

The author considers the problem of integration of the equations of motion of a solid about a fixed point under the action of a central field of force whose centre of attraction is very far from the fixed point. The equations deduced represent the generalized equations of the classical problem of motion of a solid about a fixed point. Jacobi's method of the last multiplier is used. It is shown that in the general case, the three first independent integrals exist, and, in some special cases, the fourth integral does also. The special cases are: 1) when $p=r=0$, $q=q(t)$, $\gamma'=0$, $\gamma=\gamma(t)$, $\gamma''=\gamma''(t)$, i.e., the generalized pendulum problem; 2) when $A=B$, $x_0=y_0=0$; and 3) when $x_0=y_0=z_0=0$.

D. P. Rašković (Belgrade).

Ogorodnikov, K. F. Poincaré's theorem on the upper bound of the angular velocity of rotation of stellar systems. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 38–40. (Russian)

Let a solution of the n-body problem be such that the n bodies together rotate in a rigid body motion about a fixed axis. The author applies the virial theorem to show that $\omega^2<-W/J$, where ω is the angular velocity, W is the potential energy and J is the moment of inertia about the axis. He discusses the relationship of this inequality to the analogous one of Poincaré for rotating fluid masses [H. Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, pp. 697–698].

W. Kaplan (Ann Arbor, Mich.).

Weigand, A. Die gedämpfte homogene Schwingungskette. Z. Angew. Math. Mech. **38** (1958), 28–39. (English, French and Russian summaries)

The homogeneous damped oscillatory chain consists of n equal masses with identical springs and identical viscous dampers between each pair of adjacent masses. By using difference equations and a trigonometrical representation of the complex eigenvalues, closed expressions are derived for the eigenvalues of the chain and also for the amplitudes of forced vibration and the transient response. (From the author's summary.)

G. B. Warburton.

Požarickiĭ, G. K. On the stability of dissipative systems. Prikl. Mat. Meh. **21** (1957), 503–512. (Russian)

The author considers mechanical systems with Lagrangian coordinates $q_1, \cdots, q_n$ such that $q_{k+1}, \cdots, q_n$ are cyclic; it is assumed that some of the acting forces are conservative, the rest of them derived from a dissipation function which is a positive definite quadratic form in the $\dot{q}$'s with constant coefficients. The investigation of the stability of those motions characterized by constant values of the non-cylic coordinates and of the velocities of the cyclic coordinates (not all of the latter being $=0$) is reduced to the question whether a certain quadratic form is positive definite or not.

J. L. Massera (Montevideo).

Viswanathan, K. S. The theory of the anharmonic oscillator. Proc. Indian Acad. Sci. Sect. A. **46** (1957), 203–217.

This paper chiefly considers the stable vibrations of an oscillator governed by the equation $\frac{1}{2}m\dot{x}^2+V=\bar{h}$, where $\bar{h}$ represents the total energy of the system, and $V=k_1x^2/2+k_2x^3/6$, $k_2<0$, represents the potential energy. The solution is obtained in the form of a Fourier series expansion of the Jacobian elliptic function sn, and the shift of the mean position of the oscillator, as well as the frequency and amplitude of each harmonic, is given in terms of complete elliptic integrals and the parameters of the problem. Approximate expressions, valid when $\tau=-\bar{h}^{\frac{1}{2}}k_2/6\sqrt{2}k_1^{3/2}$ is small, are calculated and found to be in agreement with earlier results. The eigenvalues of the Schroedinger equation for this oscillator are also found by the W.K.B. method.

W. E. Boyce (Troy, N.Y.).

Todeschini, Bartolomeo. Rappresentazione geometrica della labilità dei sistemi meccanici. Boll. Un. Mat. Ital. (3) **12** (1957), 596–603.

A geometric discussion is given of the locus in parameter space separating stable from unstable equilibria

of a conservative system whose potential energy depends on several parameters. *W. Kaplan* (Ann Arbor, Mich.).

Rumyancev, V. V. On the problem of motion of a heavy solid having a fixed point. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 185–188. (Russian)

A rigid body with one point O fixed is subject to gravitational force; the principal moments of inertia are assumed to be unequal; the center of mass is assumed to be distinct from O and to lie on one of the principal axes of inertia. The motion is described in terms of the Euler angles θ, ϕ, ψ; the angle ψ is cyclic and is eliminated. The differential equations for θ have stationary solutions, and conditions are obtained for their stability. The same problem was considered by Bottema [Nederl. Akad. Wetensch., Proc. **48** (1945), 316–325; MR **8**, 101], but it is asserted that Bottema's paper contains an error.

W. Kaplan (Ann Arbor, Mich.).

Lewandowski, Andrzej. An investigation of the alternating character of free non-linear vibrations by means of comparison. Arch. Mech. Stos. **10** (1958), 81–97. (Polish and Russian summaries)

A geometric method is described for determining qualitatively the alternating character of the solutions of a differential equation of the form $x''+\varphi(x, x')=0$. The method involves a comparison of the direction of the phase velocity vector in the phase plane of the system being studied with the direction of the phase velocity vector of the linear approximation of the system. Thus in a special case, by comparing the two systems, the author is able to determine a region of initial conditions starting from which the motion is non-alternating (x' vanishes at most once as the system approaches equilibrium).

J. P. LaSalle (Baltimore, Md.).

Mahalingam, S. Forced vibration of systems with non-linear, nonsymmetrical characteristics. J. Appl. Mech. **24** (1957), 435–439.

The vibrations of a single-degree-of-freedom system with non-linear elasticity under a harmonic forcing term are considered. A graphical method for finding a one-term approximate solution is described. *P. Franklin.*

Duffin, R. J.; and Schild, A. On the change of natural frequencies induced by small constraints. J. Math. Mech. **6** (1957), 731–758.

Further proofs and details are given regarding the results and techniques outlined in Proc. Symposia Appl. Math. **5** (1954), 155–163 [MR **15**, 1005]. *H. D. Block.*

Toraldo di Francia, Giuliano. Sulla traiettoria ottima di un missile leggero, soggetto a una resistenza quadratica, funzione esponenziale dell'altezza. Boll. Un. Mat. Ital. (3) **12** (1957), 401–410.

The motion of a point rocket is analysed, under the assumption that the retardation is proportional to the second power of the velocity and is an exponential function of the altitude. The weight of the rocket is assumed to be very small compared with both the thrust and the retardation. The thrust is a given function of time, but can be oriented at any instant, according to an arbitrary programme. It is asked to prescribe the direction of the thrust as a function of time, so that the horizontal range of the rocket is a maximum. It is found that under these conditions, the optimum trajectories corresponding to different thrust functions have all identical shape and can be obtained from one another by a simple translation. (From the author's summary.) *E. Leimanis.*

Leitmann, George. Trajectory programming for maximum range. J. Franklin Inst. **264** (1957), 443–452.

Necessary conditions are derived for the existence of the optimum direction of thrust and angle of attack of a rocket travelling in the atmosphere subject to forces of gravity, aerodynamic drag and lift. A mechanism is assumed capable of bringing about instantaneously the optimum angle of attack, that is, of providing the angle of attack-time function which yields maximum range between two specified altitudes. A method of solution is outlined and a singularity arising in the case of launch from zero velocity is discussed. (From the author's summary.) *J. De Cicco* (Chicago, Ill.).

See also: Ordinary Differential Equations: Colombo; Rosenberg. **Manifolds, Connections:** Stojanović. **Quantum Mechanics:** Davydov and Filippov. **Relativity:** Newman and Bergmann.

Statistical Thermodynamics and Mechanics

Kurth, Rudolf. Das Anfangswertproblem der statistischen Mechanik. J. Math. Mech. **7** (1958), 29–41.

In statistical mechanics, a system consisting of a large number of interacting particles is described by a density function in the phase space of all particles, whose time dependence is governed by the Liouville equation. This density may be projected into the subspace that corresponds to a smaller number of particles. The projected density satisfies a differential equation which, however, contains a term involving the motion of all the other particles. {This is the Yvon-Kirkwood-Born-Green-Bogolubov approach.} The author supposes that this term may be replaced by some function of the projected density alone, and gives an iteration procedure to solve the equation obtained in this way. *N. G. van Kampen.*

Tolmačev, V. V. Time correlations in classical statistical systems consisting of a large number of interaction particles. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 842–845. (Russian)

Gross, E. P.; Jackson, E. A.; and Ziering, S. Boundary value problems in kinetic theory of gases. Ann. Physics **1** (1957), 141–167.

A monatomic gas is considered to be between two infinite flat plates a distance d apart; the upper plate moves with a velocity $\frac{1}{2}w$ and the lower one with a velocity $-\frac{1}{2}w$. Stable, time independent flow is considered. The speed w is assumed to be small compared to the velocity of sound, and the mean free path is assumed to be an arbitrary multiple of d, special attention being given to the case where the mean free path is very large compared to d (Knudsen region).

There is a discontinuity in the velocity distribution function, and a method using simple half-range distribution functions is given, which is much preferred to the usual methods of expanding the distribution function in terms of full-range orthogonal polynomials.

Some properties of half-range Hermite polynomials are given in an appendix. *D. ter Haar* (Oxford).

Łunc, Michał. Détermination de la fonction de distribution des vitesses moléculaires du gaz en mouvement stationnaire par la méthode demographique. Arch. Mech. Stos. **9** (1957), 731–737. (Polish and Russian summaries)

Approximation methods are proposed for the solution of the Boltzmann transport equation in a gas in steady motion. The methods employed are entirely those of classical statistical mechanics and kinetic theory. Particular attention is paid to the effect of a bounding surface. The ideas employed might be useful in the consideration of boundary layers and adsorbed layers on surfaces. {However, in the view of the reviewer, it would be necessary to include more realistic considerations on inter-molecular and surface forces for this purpose.}

E. L. Hill (Minneapolis, Minn.).

Galkin, V. S. On a solution of the kinetic equation of Boltzman. Prikl. Mat. Meh. **20** (1956), 445–446. (Russian)

The author obtains an "exact" solution to the Boltzman equation for the "pressure tensor" in cartesian coordinates under the following conditions: (i) velocities are in the x-direction only; (ii) all variables are independent of x and z; (iii) the speed of the molecules varies linearly with y. There results a simple linear system of equations for the components of the pressure tensor. The "characteristic equation" for the system turns out to be a cubic in terms of whose roots the components of the pressure tensor may be expressed. Some simple consequences of these formulas, including limiting cases, are then discussed.

A. J. Penico (Mountain View, Calif.).

★**Yang, Hsun-Tiao; and Lees, Lester. Plane Couette flow at low Mach number according to the kinetic theory of gases.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 41–65. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

Authors apply Grad's thirteen moment approximation to solve the problem of the shearing motion between two infinite parallel plates. It is assumed that the Mach number is small with respect to unity and that the temperature difference between plates is small compared with the ambient temperature (linearization). As the result, the shearing stress is given by the usual Navier-Stokes relation (Millikan's postulate), slip flow boundary conditions are those of Maxwell (used by Millikan) and the experimental data from rotating cylinder tests at low densities agree well with this theory for $M=0.15$ to 1.40, the upper boundary of M being much higher than assumed.

M. Z. v. Krzywoblocki (Urbana, Ill.).

Gurzhi, R. N. A quantum mechanical transport equation for electrons in metals. Soviet Physics JETP **6** (1958), 352–358.

Corrections of a quantum mechanical nature to the Boltzmann transport equation are considered for electrons in a metal under the influence of an external electromagnetic field. The ultimate objective is the treatment of the anomalous behavior of the skin effect in the infra-red region of the spectrum, but the detailed application is not discussed in this paper. A modified form of density matrix is employed, collisions of the electrons with the lattice and with photons being treated with perturbation theory.

E. L. Hill (Minneapolis, Minn.).

Stratonovič, R. L. A method for the computation of quantum distribution functions. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 1097–1100. (Russian)

A procedure is suggested for the calculation of thermodynamic functions, according to the methods of quantized field theories, in which the propagator functions are used for the calculation of interactions of particles. The method is discussed for the case of a one-dimensional, non-relativistic, Bose-Einstein gas.

E. L. Hill.

Moliner, F. Garcia; and Simons, S. An extension of the general variational principle of transport theory. Proc. Cambridge Philos. Soc. **53** (1957), 848–855.

The formulation of the Boltzmann transport equation leads to difficulties when an external magnetic field is present, or when collisions with the bounding surface of the medium are taken into account [J. M. Ziman, Canad. J. Phys. **34** (1956), 1256–1273; MR **18**, 611]. The authors give an extension of the theory to cover these cases. A major point of their argument is the generalization of the definition of microscopic reversibility, which breaks down in its usual form. It is found that in the present problem, this principle can be replaced by certain other equalities connecting transition probabilities. The interpretation to be assigned to the concept of entropy production is discussed in some detail.

E. L. Hill.

Miyake, Akira. On the Kirkwood diffusion equation. Nat. Sci. Rep. Lib. Arts Sci. Fac. Shizuoka Univ. **2** (1957), no. 1, 5–11.

The article discusses formal properties of the Fokker-Planck equation for the distribution function of polymer segments which the author names the Kirkwood diffusion equation. The author discusses briefly the entropy production predicted by this equation, its spectral operator solution, and the relaxation function in terms of this operator L.

J. Ross (Providence, R.I.).

Bellemans, A. La tension superficielle des solutions déduite directement de l'intégrale de configuration. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 663–668.

The theory of conformal solutions, introduced by Longuet-Higgins with the aid of a generalization of the theorem of corresponding states, is based on the assumption that the energy of interaction of two species, r and s, separated by a distance ρ is given by $u_{rs}(\rho)=f_{rs}u_{00}(g_{rs}\rho)$, where u_{00} is the energy of interaction of two molecules of the reference specie 0, and f_{rs} and g_{rs} are parameters. The author extends the expression for the configurational Gibbs free energy as derived by Longuet-Higgins and Brown to systems with surfaces, and obtains a general equation for the excess surface tension $\sigma^E=\sigma-x_1\sigma_1-x_2\sigma_2$ for a solution. An approximate but explicit expression is obtained for σ by neglecting order-disorder terms and evaluating fluctuation terms for a simple model.

J. Ross (Providence, R.I.).

See also: Probability: Scheidegger. **Elasticity, Plasticity:** Eirich; Nakada. **Quantum Mechanics:** Kirzhnits; Van Hove; Temkin; Ladányi.

Elasticity, Plasticity

Doyle, Thomas C. Higher order invariants of stress or deformation tensors. J. Math. Phys. **36** (1958), 297–305.

The author derives a fundamental system of irrotational

scalar differential invariants, of arbitrary fixed order p, of a Euclidean metric tensor and another symmetric second order tensor a for $p=1$; an equivalent rotational system is obtained. A neat and well-documented treatment of known necessary and sufficient conditions for the existence of isotropic surfaces is given. Simplifications permitted when a satisfies certain algebraic and differential equations are pointed out. The author suggests that these results may be useful in constructing generalizations of nonlinear elasticity theory. *J. L. Ericksen.*

Rogoziński, Marian. On the possibility of better utilization of the analogy between stress and strain in the theory of elasticity. Arch. Mech. Stos. **9** (1957), 713–730. (Polish and Russian summaries)

The author attempts to show by example that the solution of problems in linear elasticity is facilitated by introducing a vector V such that the symmetric part of ΔV equals the stress. He fails to convince the reviewer. The stress must satisfy compatibility conditions mentioned by the author for V to exist. *J. L. Ericksen.*

Nicolovius, Rüdiger. Beiträge zur Diaz-Greenberg-Methode. Z. Angew. Math. Mech. **37** (1957), 449–457. (English, French and Russian summaries)

The author uses the hypercircle method introduced by Prager and Synge [Quart. Appl. Mech. **5** (1947), 241–269; MR **10**, 81] to show that the Diaz-Greenberg method [ibid. **6** (1948), 326–331; J. Math. Phys. **27** (1948), 193–201; MR **10**, 167, 213] can be applied to a fairly general boundary value problem involving a fourth order differential equation to obtain bounds on the solution at a point interior to the domain of definition of the problem.

Appropriate inequalities of the Diaz-Greenberg type are also given for a self-adjoint elliptic boundary value problem of second order, a parabolic boundary value problem of fourth order and boundary value problems of elasticity.

In a comparison based on a plate problem, it appears that this method gives more accurate results but requires more work than the hypercircle method. *C. G. Maple.*

Truesdell, C. Geometric interpretation for the reciprocal deformation tensors. Quart. Appl. Math. **15** (1958), 434–435.

Hill, R. On uniqueness and stability in the theory of finite elastic strain. J. Mech. Phys. Solids **5** (1957), 229–241.

This self-contained paper establishes a direct relation between the criterion of stability of an elastic solid in a state of finite strain and the criterion for the existence of a unique solution of the associated boundary value problem for given increments of displacement and tractions on the surface of the solid. A connexion is shown to exist between the structure of these criteria and the concept of functional convexity in relation to the strain-energy density. The limit of stability is characterized by eigenfunctions or adjacent positions of equilibrium.

The author refers, in particular, to the comprehensive paper by Pearson [Quart. Appl. Math. **14** (1956), 133–144; MR **18**, 82]. Some earlier important contributions [Trefftz, Z. Angew. Math. Mech. **13** (1933), 160–165; Kappus, ibid. **19** (1939), 344–361; MR **1**, 188] are missing in the list of references. Eqs. (20) to (22) of the present paper are essentially equivalent to Trefftz' equations, and the equivalence of the energy criterion of stability and the existence of adjacent positions of equilibrium were discussed extensively by Kappus; the latter discussion effectively contained also the connexion between uniqueness and stability. *W. T. Koiter* (Delft).

Atsumi, A. On the stresses in a strip under tension and containing two equal circular holes placed longitudinally. J. Appl. Mech. **23** (1956), 555–562.

This paper deals with the plane problem of elasticity (characterized by the biharmonic stress function) for two equal circular holes placed longitudinally on an infinite strip under tension, where polar coordinates are employed. The solution is obtained with the aid of a perturbation method, but the convergence of the series involved is not established. Detailed numerical examples are given. *P. M. Naghdi* (Berkeley, Calif.).

Tamate, Osamu. The effect of a circular hole on the pure twist of an infinite strip. J. Appl. Mech. **24** (1957), 115–121.

The effect of a circular hole in an infinite strip subjected to a state of pure twist at infinity is studied with the use of polar coordinates and with the aid of a puerturbation technique. The maximum value of the stress couple at the periphery of the hole is presented in series form, and numerical results are given; the convergence of the series, however, is not established. *P. M. Naghdi.*

Narodetsky, M. Z. The solution of two-dimensional problems in the theory of elasticity by means of special functions. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 729–732. (Russian)

Complex variable treatment of plane stress in an infinite plate with two circular holes, boundaries L_1, L_2. The special functions α_n, β_n mentioned in the title, are defined by recurrence relations

$$\alpha_n(z)=\frac{\xi_{n-1}}{2\pi i}\int_{L_1}\frac{\overline{\beta_{n-1}(t)}}{t-z}\,dt,\quad \beta_n(g)=\frac{\xi_{n-1}^*}{2\pi i}\int_{L_2}\frac{\overline{\alpha_{n-1}(t)}}{t-z}\,dt,$$

where ξ_n, ξ_n^* are constants.

Detailed calculations are carried out for the case of a concentrated force applied at the point on L_1 nearest to L_2 and directed along the line of centres, the two circles having equal radii. *R. C. T. Smith* (Armidale).

Nowacki, Witold. A plane distortion problem. Arch. Mech. Stos. **9** (1957), 417–438. (Polish and Russian summaries)

This paper deals with a simply connected thin isotropic plate, free from edge stress, under a given state of initial strain over a region of the plate. The strain and stress throughout the plate are determined. The results involve the Green's functions for the plate, which have been determined in the case of (i) an infinite plate, (ii) a semi-infinite plate, and (iii) a plate strip. *R. M. Morris.*

Stippes, M. Displacements in plane elasticity. Z. Angew. Math. Mech. **38** (1958), 71–72.

Assuming that the stress at any point (x_0, y_0) in rectangular cartesian coordinates may be developed in a Taylor series expansion and, further, that the rigid body motion of the element in the neighbourhood of this point is given, then general expressions are found for the displacements at any other point (x, y) in plane elasticity.

R. M. Morris (Cardiff).

Szelągowski, Franciszek. Solution of the plane problem of elasticity in a system of Cartesian coordinates, mass forces being taken into consideration. Arch. Mech. Stos. **10** (1958), 99–105. (Polish and Russian summaries)

The basic equations of equilibrium of plane elasticity in cartesian form are solved in terms of harmonic functions for the case when the mass forces are not zero. The particular form of the functions representing the mass forces is discussed and the particular case when these forces have a potential is treated. *R. M. Morris.*

★ **Mohan, Madan. On stresses in a thin isotropic elastic plate in the form of epitrochoid rotating steadily in its plane.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 95–98. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

Stresses due to uniform rotation of a thin elastic plate in the form of epitrochoids have been determined. The region is transformed into the interior of a unit circle and by an application of Muskhelishvili's method the problem has been reduced to the solution of integral equations of which a function-theoretic solution has been obtained. [Author's abstract.] *R. M. Morris* (Cardiff).

★ **Roy, S. K. On the biharmonic analysis of stresses round openings in structures in relation to unlimited stress fields.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 57–74. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

In the treatment of non-cyclic stress fields in multiply connected structures, the Mitchell conditions for non cyclicity of the displacements and rotations on a free internal boundary are the determining criteria for the nature of the additional stress fields which are superposed on the initial field due to the existence of the internal boundary. For the biharmonic analysis of finite structures under stresses with such internal openings, the numerical analysis does not have to encounter the infinities associated with the unlimited stress field, when a vanishing stress field at infinity does not necessarily imply a vanishing or finite stress function. The analytical treatment of the infinities in this connection is examined, and the methods to be adopted for analysis worked out so as to avoid the arbitrariness in the adoption of the extent of field required for analyzing the stresses without prejudice to the use of the Saint Venant Principle. Results are given for the circular opening for which solution in closed form exists, and for the square opening, for which the stress reversals and concentrations and typical isochromatics are worked out. The method is important for a correct analysis of openings in unlimited stress fields, and useful for the design requirements of concentrations and stress reversals in conduits and galleries in structures. [Author's summary.] *R. M. Morris* (Cardiff).

Hardiman, N. J. Two-dimensional problems in elasticity involving different media. Proc. London Math. Soc. (3) **7** (1957), 584–597.

Using complex potential methods, some problems of isotropic elastic plates in generalised plane stress condition, including inclusions of a second isotropic elastic material, have been solved. The author has considered the following boundary-value problems: A) An unstressed hole in an infinite plate with an annular reinforcement when the plate is under 1) an all-round tension τ at infinity, 2) a simple tension τ at infinity; B) a circular hole in an infinite plate with an annular reinforcement, the inner boundary being 1) under uniform pressure and/or uniform shear, 2) under given edge stresses, 3) under given edge displacements.

Lastly, the special case of a core of rigid material of region $0 \leqq r \leqq b$, which adheres completely to the elastic material of region 1 on the boundary $r=b$, has been considered, when the plate and the core are under different conditions of stress and position respectively.

The results are expressed in terms of the following effective elastic constant combinations:

$$\alpha=\frac{E_1}{E_2},\quad \gamma=\tfrac{1}{8}E_1\left(\frac{1}{\mu_2}-\frac{1}{\mu_1}\right),$$

where μ_1 and μ_2 are the rigidities. *W. Zerna.*

Sherman, D. I. On a problem in the theory of elasticity with mixed homogeneous conditions. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 733–736. (Russian)

The author considers a finite region S in a complex plane $z=x+iy$ bounded by a simply connected and sufficiently smooth curve L with the origin inside S. The normal component of the displacement vector and the tangential component of the stress vector on the boundary L are prescribed. The problem is to find the displacement vector and the stress tensor inside S. The author has previously investigated a more general problem [Prikl. Mat. Meh. **7** (1943), 413–420; MR **6**, 195], in which the region was multiply-connected. The solution of this more general problem led to a system of Fredholm integral equations which were very difficult to solve because the kernels themselves were integrals not expressible in a closed form. A simply connected region simplifies the problem, and the author obtains a system of Fredholm integral equations with kernels expressible in terms of elementary functions. The method of analytic functions is used throughout. *T. Leser* (Aberdeen, Md.).

Ling, Chih-Bing. Stresses in a perforated strip. J. Appl. Mech. **24** (1957), 365–375.

Stresses are obtained in an infinite strip having an eccentric circular hole and subject to stresses symmetric about the line of symmetry. The solution valid throughout the strip is given in terms of one biharmonic integral and four biharmonic series; the series is formed from a class of periodic functions defined by

$$W_s(z)=\frac{(-i)^s}{(s-1)!}\frac{d^s}{dz^s}W_0(z),$$

where $W_0(z)=-\log\{\sinh\frac{1}{2}\pi z\ \sinh\frac{1}{2}\pi(z-2i+2ic)\}$. Two numerical examples of longitudinal tension and transverse bending are worked out in detail. A full discussion of the function $W_s(z)$ is given in an appendix. *B. R. Seth.*

Schaefer, Hermann. Die vollständige Analogie Scheibe – Platte. Abh. Braunschweig. Wiss. Ges. **8** (1956), 142–150.

Die Wieghardtsche Analogie zwischen Airyscher Spannungsfunktion der Scheibe und der Durchbiegung einer Platte wird vervollständigt durch Einführung von zwei Spannungsfunktionen der Platte, die den Verschiebungen der Scheibe entsprechen. *Zusammenfassung des Autors.*

Ivanov, C. I. On the solution of plane elasticity problems by finite differences. Akad. Nauk Ukraïn. RSR. Prikl. Meh. 3 (1957), 339–344. (Ukrainian. Russian summary)

Rvačev, V. L. On the pressure exerted by a plane die on an elastic half-space. Prikl. Mat. Meh. 21 (1957), 444–445. (Russian)

The author considers an elastic half-space compressed by a die. The contact region S between the space and the die is bounded by a second degree curve. The problem consists of finding a harmonic function $w(x, y, z)$ which becomes a constant at all points of the contact region S. The author constructs a function $F(x, y, z, \lambda, \alpha, \beta)=0$ representing a family of level surfaces for the function w; here λ is a parameter depending on w, and α and β are constants. Applying the Laplace operator and setting $\partial^2 w=0$, the author obtains an ordinary second order differential equation for $\lambda(w)$. Solution of this last equation yields w. Depending on the values of α and β, the boundary of the contact region is an ellipse, hyperbola, or two lines. The author does not explain how he derived or guessed the function F, saying only that it turns out that way for this particular case. *T. Leser* (Aberdeen, Md.).

Meškov, A. I. Equilibrium of an elastic parallelepiped. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. 12 (1957), no. 2, 35–43. (Russian)

Having an inclined parallelepiped, the author introduces a system of rectilinear skew coordinates $Oxyz$, where the coordinate axes are parallel to the edges of the parallelepiped. Using tensor notation, he derives conversion from a rectangular to a skew coordinate system, and writes down expressions for the potential energy of a unit volume and of the whole body. In all cases the author uses the variational principle of Castigliano. He considers the following two cases: 1) torsion with two different kinds of load, and 2) thermal stresses caused by temperature $t(x, y, z)$, in the absence of external loads. *T. Leser.*

Vlasov, B. F. On a case of bending of a thick rectangular slab. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. 12 (1957), no. 2, 25–34. (Russian)

The author solves the elasticity problem of a thick plate for very special boundary conditions. The dimensions of the plate are a_0, b_0, h_0; the displacement at a point (x, y, z) are (u, v, w); and the boundary conditions are as follows: at $x=0$ and at $x=a_0$: $w=v=\sigma_x=0$; at $y=0$ and at $y=b_0$: $w=u=\sigma_y=0$; at $z=h_0/2$: $\tau_{xy}=\tau_{yz}=\sigma_z=0$; at $z=-h_0/2$: $\tau_{xy}=\tau_{yz}=0$, $\sigma_z=-q(x, y)$; where the distributed load $q(x, y)$ is given in a form of a trigonometric series. The author solves the equilibrium equations by separation of variables, which reduces partial differential equations to ordinary differential equations and finally to a large system of algebraic linear equations. He compares his results with the results obtained from the approximate theory (Sophie Germain) for thin plates and, except for σ_z, which may differ from the correct result up to 16 percent, the approximate theory is amazingly good. *T. Leser.*

Steketee, J. A. On Volterra's dislocations in a semi-infinite elastic medium. Canad. J. Phys. 36 (1958), 192–205.

The dislocation considered is caused by a discontinuity of displacement of amount $\mathbf{U}+\mathbf{\Omega}\times\mathbf{r}$, where $\mathbf{U}$ and $\mathbf{\Omega}$ are constant vectors specified by 6 constants, across a surface S. The resulting displacement in an infinite elastic medium is equivalent to that due to a distribution over S of nuclei of strain which arise from isolated single forces or from combinations of such forces (double forces, with or without moments). Formulae for the displacement in a semi-infinite medium due to typical nuclei of strain are given, and that due to a particular double force system is calculated in detail. The displacement from the dislocation can then be expressed as an integral over S involving the 6 constants. *W. R. Dean* (London).

Solyanik-Krassa, K. V. On the problem of elastic equilibrium of rotational bodies. Dokl. Akad. Nauk SSSR (N.S.) 114 (1957), 49–52. (Russian)

For displacements $u=u_m(r, z)\cos m\phi$, $v=v_m(r, z)\sin m\phi$, $w=w_m(r, z)\cos m\phi$ satisfying the equations of equilibrium in cylindrical polar coordinates, $u_m(r, z)$, $v_m(r, z)$, $w_m(r, z)$ are expressed in terms of three functions f_m, φ_m, χ_m such that

$$\left(\frac{\partial^2}{\partial r^2}-\frac{2m+1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\begin{Bmatrix} f_m \\ \varphi_m \\ \chi_m \end{Bmatrix}=0.$$

R. C. T. Smith (Armidale).

Chattarji, P. P. Stresses in a spherically isotropic truncated cone fitted with two rigid spherical caps due to frictional forces acting on the curved surfaces. Indian J. Theoret. Phys. 5 (1957), 15–18.

Stresses in a spherically isotropic truncated cone depressed by two rigid spherical caps and acted upon by uniform shearing stress on the curved surface are obtained in terms of conical harmonics. *B. R. Seth.*

Dutt, S. B. Stresses in an aeolotropic paraboloid of revolution due to frictional force acting on its surface. Indian J. Theoret. Phys. 5 (1957), 11–14.

Displacements in an aeolotropic paraboloid of revolution subject to uniform shearing stress on the curved surface are obtained in terms of Bessel functions of first kind with the help of the mapping $z+ir=(\xi+i\eta)^2$. *B. R. Seth* (Kharagpur).

★ **Jain, M. K. Problems of cross-elasticity.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 81–86. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

Using an approximate stress-finite strain relation, the author solves problems of (i) simple tension, (ii) hydrostatic pressure, (iii) longitudinal vibrations of an isotropic elastic body. *A. E. Green.*

Adkins, J. E. A three-dimensional problem for highly elastic materials subject to constraints. Quart. J. Mech. Appl. Math. 11 (1958), 88–97.

The author considers a body constrained so that it is inextensible in three directions which may be taken as coordinate lines of a curvilinear coordinate system. He characterizes the most general form of a deformation consistent with these constraints when these coordinate surfaces are initially plane. He shows that presumption of the displacement over a surface, plus the constraints, uniquely determines all displacement gradients over the surface except in the case of coordinate surfaces. The analysis given is purely kinematical, though the general equations of nonlinear elasticity are introduced and discussed in some detail. Adequate references to related work are included. *J. L. Ericksen* Baltimore, Md.).

Blackburn, W. S. Second order effects in the flexure of isotropic incompressible elastic cylinders. Proc. Cambridge Philos. Soc. **53** (1957), 907–921.

The general theory of Green and Spratt [Proc. Roy. Soc. London. Ser. A. **224** (1954), 347–361; MR **16**, 89] for second order effects in the theory of elasticity for isotropic incompressible materials is applied to problems of prismatic beams in bending under transverse end loads (Saint Venant's problem in the classical theory). Complex variable technique is used, and the stresses and displacements are obtained in terms of classical flexure, torsion functions and eight additional complex potential functions. All functions are obtained explicitly for the case of a circular cylinder under eccentric transverse end load. *W. T. Koiter* (Delft).

Polozhy, G. N. On certain overall characteristics of the stressed state of prismatic rods acted upon by a bending force. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 45–48. (Russian)

A simple method, based on a property of subharmonic functions, is given for estimating the maximum shearing stresses in a bar bent by a terminal load. *R. C. T. Smith.*

Amenzade, Yu. A. The bending of a prismatic rod weakened by a circular cavity. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 37–40. (Russian)

A complex variable solution is given of the bending problem for a bar whose cross-section is the region between the "square with rounded corners"

$$Z=A\left(\zeta-\frac{1}{9\zeta^3}\right),\quad |\zeta|=1,$$

and the enclosed circle $|z|=R$. *R. C. T. Smith.*

Položiĭ, G. M. On some comparison theorems for boundary problems of torsion of prismatic bars. Akad. Nauk Ukrain. RSR. Prikl. Mat. **1** (1955), 391–399. (Ukrainian. Russian summary)

Let G be the simply connected plane cross section of a prismatic bar in torsion; L be the smooth simple closed curve which is the boundary of G; M be the twisting moment; D be the torsional rigidity; τ be the angle of twist per unit length of the bar; and $\mathbf{V}=X_z+iY_z$ be the stress vector. Consider also a similar prismatic bar with cross section G_1 contained in G, and let the corresponding quantities relative to G_1 be designated by letters with the subscript one: Theorem 1: If $M=M_1$, then along the common part LL_1 of the boundaries of G and G_1, one has $D_1|\mathbf{V}_1|\leq D|\mathbf{V}|$. Theorem 2: If $\tau=\tau_1$ then $|\mathbf{V}_1|\leq|\mathbf{V}|$ on LL_1. Theorem 3: $D_1\leq D$. An application of these inequalities is made to a centrally symmetric cross section bounded by four circular arcs. *J. B. Diaz* (College Park, Md.).

★ **Sharma, Brahmadev. Stresses due to shearing forces on a plane boundary of an infinite elastic slab with transverse isotropy.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 75–80. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

The author solves the problem of an infinite transversely isotropic elastic slab with the lower plane boundary fixed and the upper plane boundary subjected to a torsional shear load and, in particular, a concentrated couple. This is accomplished by superposing two solutions. The first solution is for a semi-infinite body subjected to the same torsional boundary loading as on the given slab. The second solution annuls the non-zero displacement in the semi-infinite body of the first solution which occurs at a distance below the boundary plane equal to the given slab thickness. In the isotropic case, the first solution for a semi-infinite body becomes identical with that given earlier by the reviewer [Quart. J. Mech. Appl. Math. **7** (1954), 287–298; MR **16**, 309]. *Yi-Yuan Yu.*

Dutt, S. B. Stress concentrations around a small inclusion formed by the inverse of a planetary ellipsoid with respect to its centre on the axis of a circular cylinder in torsion. J. Assoc. Appl. Phys. Calcutta Univ. **4** (1957), 53–58.

Stresses are obtained around a rigid inclusion, in the form of an inverse of a planetary ellipsoid with respect to its centre on the axis of a circular cylinder under torsion, in terms of Legendre polynomials of the first and second kind with the help of curvilinear coordinates introduced by $r+iz=c\sec(\xi+i\eta)$. No numerical results are given. There are a number of misprints. *B. R. Seth.*

Rózsa, Pál. Die Anwendung des Matrizenkalküls auf die Statik von Balken und Platten. Magyar Tud. Akad. Mat. Kutató Int. Közl. **1** (1956), 593–621 (1957). (Hungarian. Russian and German summaries)

Ziel der Untersuchung ist die Berechnung der vertikalen Deformation von (senkrecht und) kontinuirlich belasteten Balken und rechteckigen Platten. Bekannterweise führen diese Probleme zu den Differentialgleichungen (1) $y^{(4)}=p(x)/EJ$ und (2) $\Delta\Delta w=t(x,y)/N$, wobei $p(x)$ und $t(x,y)$ die kontinuirlich verteilten Belastungen und beim Balken E den Youngschen Modul und J das Flächenmoment des Balkenquerschnittes bedeutet. Eine analoge Bedeutung hat in der Gleichung der Platte N. (Δ ist hier der Laplacesche Operator im zweidimensionalen Fall.) Die gewohnte Methode der Lösung dieses Problems ist z.B. beim Balken die, dass die Differentialgleichung zuerst für eine in einem Punkte angreifende Kraft analytisch (unter Berücksichtigung der speziellen Randbedingungen) gelöst wird. $y(x)$ ist dann die sogenannte Greensche Funktion (Einflussfunktion) des Problems. Diese Funktion muss im Falle von kontinuirlich verteilten Belastungen über diese integriert werden. Der Verfasser ersetzt dagegen die Differentialgleichungen (1) und (2) durch Systeme von Differenzengleichungen, benutzt also die sogenannte Gittermethode, wobei die verteilten Belastungen durch konzentrierte ersetzt werden. Diese Differenzengleichungen sind Systeme von linearen algebraischen Gleichungen; von den Randbedingungen werden z.B. beim Balken nur die erste und die letzte unmittelbar beeinflusst. Das ganze zu lösende mathematische Problem, das ausführlich besprochen wird, ist eigentlich die Berechnung der reziproken Matrix des erwähnten Gleichungssystems, die ein finites Analogon der Greenschen Funktion ist. Tatsächlich erhält man aus den Elementen dieser Matrix durch Grenzübergang die Greenschen Funktionen. Behandelt werden der an beiden Enden unterstützte, der an ebenfalls beiden Enden eingespannte und der nur an einem Ende eingespannte Balken, ausserdem die an zwei gegenüberliegenden Seiten unterstützte und an den anderen zwei eingespannte, sowie die an allen vier Seiten unterstützte und eingespannte Platte. Besonders das letztere Problem verursacht mathematische Schwierigkeiten, weil bei dem die Koeffizientenmatrix aus nicht vertauschbaren Blöcken besteht. Dieses Problem wird ausführlich besprochen. *Th. Neugebauer* (Budapest).

Pisarenko, G. S. Die Querschwingungen der Stäbe mit Erwägung der Materialverluste. Apl. Mat. **2** (1957), 424–443. (Czech. Russian and German summaries)

The author presented his theory of vibrations in a dissipative medium and applications of it in his Russian book, "Vibrations of elastic systems taking account of dissipation of energy in the material" [Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1955; MR **19**, 195]. This paper deals with forced and free transverse vibrations of a rod with a constant cross-section. The treatment of forced vibrations in this paper is similar to the one given in the second chapter of the above mentioned book. Dissipative losses are assumed to be proportional to the amplitude. Material which is considered has a narrow hysteresis loop which means that both branches of the loop differ little from straight lines of Hooke's law, hence the author is justified in introducing a small parameter in his non-linear differential equations. He solves approximately these equations using the Krylov-Bogoliubov method of asymptotic expansion in powers of small parameter, solving separately the case of forced vibrations and of free vibrations.

T. Leser (Aberdeen, Md.).

Schaefer, Hermann. Die Spannungsfunktionen einer Dyname. Abh. Braunschweig. Wiss. Ges. **2** (1954), 107–112.

Es wird der Tensor der Spannungsfunktionen eines Stabes untersucht, der durch eine Dyname belastet ist. Zusammenhänge mit der Volterraschen Theorie der Distorsionen werden besprochen.

Zusammenfassung des Autors.

Günther, Wilhelm. Über das Gleichgewicht an einer randbelasteten Schale. Abh. Braunschweig. Wiss. Ges. **8** (1956), 111–120.

Die allgemeine Lösung der sechs Gleichgewichtsbedingungen in der Biegetheorie der Schalen enthält sechs willkürliche Funktionen. Es wird gezeigt, daß diese zu zwei Vektoren zusammengefaßt werden können, den "Vektoren der Spannungsfunktionen". Für verschwindende Momente um die Schalennormale hängen diese Vektoren noch von vier willkürlichen Funktionen ab; in der Membrantheorie genügt die verbleibende eine Spannungsfunktion einer Differentialgleichung zweiter Ordnung. Diese Differentialgleichung wurde zuerst von Lagally aufgestellt. *Zusammenfassung des Autors.*

Muštari, H. M. On inverse boundary problems in the non-linear theory of curved shells. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 35–37. (Russian)

Bulgakov, V. N. Toroidal shell under the action of centrifugal forces. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 215–224. (Ukrainian. Russian and English summaries)

V. V. Novožilov's complex theory of thin shells [Theory of thin shells, Gos. Izdat. Sudostroitel. Lit., Moscow, 1951; MR **17**, 915] is applied to a rapidly rotating toroidal shell filled with a fluid. The special functions occurring in the solution are tabulated. The effect of boundary conditions on the stresses is considered. *R. C. T. Smith.*

Kacner, Artur. A closed solution in the case of a semi-infinite plate with discontinuous boundary conditions. I. Arch. Mech. Stos. **9** (1957), 371–380. (Polish and Russian summaries)

For a semi-infinite strip with the edges $x=0$, $x=a$ hinged, solutions are known for the following problems: (i) Uniform normal pressure applied to plate, the edge $y=0$ being free; (ii) Concentrated unit moment parallel to the x-axis applied at $x=z$, $y=0$. This paper is concerned with a semi-infinite strip under uniform normal load with boundary conditions at $y=0$, $w=\omega M(x)$, $T(x)=0$ for $0\leqq x\leqq d$, $M(x)=0$, $T(x)=0$ for $d\leqq x\leqq a$, where $M(x)$ is the bending moment, $T(x)$ the shearing force per unit length and ω the Winklerian support coefficient. Using the results mentioned above, this problem is reduced to a Fredholm integral equation of the second kind, which is then solved in closed form. The limiting behaviour of the solution as $\omega\to 0$ is investigated. *R. C. T. Smith.*

★ **Joga Rao, C. V. Long rectangular plates subjected to linearly varying loads.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 8–15. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

The paper considers large deflections of a long rectangular plate subjected to lateral pressure loading which increases linearly from one edge to the other in the shorter direction. Equations are given for a plate with simple supported edges, and also for a plate with fixed edges. The results are given in a form similar to those in Timoshenko's book, "Theory of plates and shells" [McGraw-Hill, New York, 1940, Ch. 1] for uniformly loaded rectangular plates. It is concluded that the design of plates with uniformly varying load can be based on an equal total load uniformly distributed. The linear variation in load in the cases studied gave results almost identical to those for uniform loading. *S. Levy.*

Nariboli, G. A. Mixed boundary value problems for rectilinear plates. I. Arch. Mech. Stos. **9** (1957), 507–524. (Polish and Russian summaries)

W. Nowacki's method [Arch. Méc. Appl., Gdańsk **3** (1951), 419–435; Arch. Mech. Stos. **5** (1953), 193–220; MR **14**, 601; **16**, 310] is used to solve the problem of the bending of thin plates, partly supported and partly clamped on the contour. The Green's function is obtained by making use of analogies for a simply supported plate. A concentrated load corresponds to a vortex and a uniformly distributed load to the torsion problem. The mixed problem for a rectilinear plate is then reduced to the solution of a Fredholm integral equation of the first type. The cases of a right angled isosceles plate and one with 30°-60°-90° angles are discussed in detail.

B. R. Seth (Kharagpur).

Conway, H. D. On an axially symmetrically loaded circular shell of variable thickness. Z. Angew. Math. Mech. **38** (1958), 69–70.

Bereits Reissner und kürzlich auch Federhofer haben den Fall einer in der Längsrichtung veränderlichen Wandstärke bei axialsymmetrisch belasteten Kreiszylinderschalen behandelt, wobei der erstere ein lineares, der letztere ein parabolisches Änderungsgesetz zugrunde legte. Federhofer zerlegte die übliche Differentialgleichung 4. Ordnung für die Radialverschiebung w (auch als Behältergleichung bekannt) in zwei simultane Gleichungen, die er durch hypergeometrische Funktionen löste. Für den Fall, daß die Wandstärke sich mit dem Abstandsquadrat von einem beliebigen Ursprung ändert, läßt sich die Lösung durch elementare Funktionen ausdrücken.

Conway weist nun nach, daß man in diesem Sonderfall

viel einfacher zum Ziel gelangt, wenn man von der üblichen Differentialgleichung 4. Ordnung ausgeht. Für den oben erwähnten Sonderfall geht nämlich die an sich für veränderliche Wandstärken recht schwierige Differentialgleichung mit variablen Koeffizienten in eine gleichdimensionale (homogene) Gleichung über, die man durch Substitution in eine Differentialgleichung mit konstanten Koeffizienten überführen kann. Diese läßt sich leicht lösen, wobei sich insbesondere die Wurzeln der charakteristischen Gleichung gut ziehen lassen. *W. Zerna.*

Reissner, Eric; and Sledd, M. B. Bounds on influence coefficients for circular cylindrical shells. J. Math. Phys. **36** (1957), 1–19.

This paper is concerned with rotationally symmetric deformation of a thin elastic circular cylindrical shell of variable wall thickness and of finite or semi-infinite axial length; one of the two edges of the shell is free, and the other, which is acted upon by a uniformly distributed bending couple M_0 and a radial force H_0, undergoes a radial displacement u_0 and a rotation β_0. With the assumption of linearity between stresses and strains, it follows that $u_0=-c_{uH}H_0+c_{uM}M_0$, or $H_0=-k_{uH}u_0-k_{\beta H}\beta_0$ (and similar expressions for β_0 and M_0), where c and k are called "direct influence coefficients" and "inverse influence coefficients", respectively. The authors establish the upper and lower bounds for these coefficients by undertaking the "bounding of several such coefficients simultaneously"; this approach involves considerations which do not arise when only one influence coefficient is present in a given problem. As an example of the application of the inequalities derived, the case of a semi-infinite shell with exponentially varying wall thickness is considered in detail. *P. M. Naghdi* (Berkeley, Calif.).

Reissner, Eric. Finite twisting and bending of thin rectangular elastic plates. J. Appl. Mech. **24** (1957), 391–396.

An exact solution of the nonlinear differential equations for finite deflection of thin elastic plates is obtained for a rectangular plate under the simultaneous action of twisting moments T and bending moments M on two opposite edges. Relations are deduced between T and M on the one hand, and between the angle of twist per unit length θ and the curvature $\varkappa$ on the other. These relations represent a generalization of previously known results when either $\varkappa=0$ or $\theta=0$. *P. M. Naghdi* (Berkeley, Calif.).

★**Hammad, H. Y. Approximate theory of bending of short cylindrical shell roofs.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 1–7. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

An approximate solution is obtained for the stresses in segments of circular cylindrical shells which are simply supported along meridional boundaries, free along longitudinal boundaries, and subject to uniform lateral load. A membrane solution of this problem does not satisfy the prescribed boundary conditions along the free edges of the shell. The author obtains a bending solution for the case that the distance between the meridional boundaries is small compared with the distance between the longitudinal boundaries by assuming that meridional bending moment and twisting moment are negligibly small compared with the longitudinal bending moment. The solution of the resultant approximate system of shell equations predicts that deviations from the results of membrane theory are confined to narrow edge zones along the free longitudinal boundaries. The theoretical results are illustrated by a numerical example. *E. Reissner.*

Peštmaldžyan, D. V. On the calculation of symmetrically loaded laminar anisotropic shells of revolution. Izv. Akad. Nauk Armyan. SSR. Izv. Ser. Fiz.-Mat. Nauk **10** (1957), no. 2, 39–54. (Russian. Armenian summary)

The author considers a thin shell built of an arbitrary number of orthotropic layers. The planes of elastic symmetry of each layer are parallel to each other. The outside concave surface is chosen as the coordinate surface, and the coordinate axes coincide with the lines of principal curvatures of the surface. The solution of the problem of elasticity for such a shell can be obtained from a system of differential equations given by S. A. Ambarcumian [same Izv. **6** (1953), no. 3, 15–35]. The author solves this system for cylindrical and conical shells. He gives one numerical example for a semi-infinite two-layered cylindrical shell with a specified ring-shape distributed load and with a Poisson ratio of zero.

T. Leser (Aberdeen, Md.).

Morozov, N. F. On the non-linear theory of thin plates. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 968–971. (Russian)

Existence theorems for the system of non-linear equations controlling large deflections of thin plates have been given before by a number of investigators. The author of this paper presents a different treatment of this problem, one using the functional analysis.

Attention should be drawn to the author's notation and to a certain mistake. In the system of partial differential equations for large deflections of plates, the author uses the symbol Δ for the Laplacian, $\partial^2/\partial x^2+\partial^2/\partial y^2$; hence the symbol Δ^2 in the author's notation stands for $(\partial^2/\partial x^2+\partial^2/\partial y^2)(\partial^2/\partial x^2+\partial^2/\partial y^2)$. In the second equation of the system, marked by the author as (1), the left member should be $\Delta^2 w$ instead of $\Delta^2\omega$. The paper is not very well organized; for example, one has to guess in the beginning that S stands for the boundary and Ω for the domain, since these two symbols are not defined. *T. Leser.*

Ambarcumyan, S. A. On a problem in the non-linear theory of anisotropic plates. Akad. Nauk Armyan. SSR. Dokl. **24** (1957), 153–159. (Russian. Armenian summary)

The author considers a plate of constant thickness h, the coordinate plane $\alpha\beta$ coinciding with the middle plane, and the coordinate axis γ directed toward the loaded boundary plane. The material obeys the generalized Hooke's law, and at every point the plane of elastic symmetry is parallel to the coordinate plane $\alpha\beta$. The author assumes that: a) elements normal to the middle plane retain their length after the deformation; b) the strains ε_α, ε_β and $\varepsilon_{\alpha\beta}$ could be determined by neglecting the stress σ_γ; c) when determining the strains $\varepsilon_{\alpha\gamma}$ and $\varepsilon_{\beta\gamma}$, the stresses $\sigma_{\alpha\gamma}$ and $\sigma_{\beta\gamma}$ could be considered equal to the corresponding stresses obtained on the assumption that normals remain undeformed; d) when determining deformations of the middle surface, all non-linear terms could be neglected except those arising from the normal displacements w. Using the above assumptions, the author modifies the non-linear system of equations controlling anisotropic plates as given by S. G. Lehnickiĭ [Anisotropic plates,

OGIZ, Moscow-Leningrad, 1947; MR **10**, 415], and solves the above-mentioned system for a rectangular plate with specific boundary conditions and a constant distributed load. *T. Leser* (Aberdeen, Md.).

Ambarcumyan, S. A. On two methods of analysing two-layered orthotropic shells. Izv. Akad. Nauk Armyan. SSR. Izv. Ser. Fiz.-Mat. Nauk **10** (1957), no. 2, 17–38. (Russian. Armenian summary)

The author considers a thin two-layered shell, each layer being made of a homogeneous orthotropic material. The planes of elastic symmetry of each layer are perpendicular to each other. The surface of contact between the two layers was chosen as the coordinate surface; the coordinate axes α and β are lying on this surface, the coordinate axis γ is normal to it. The layers remain elastic after the deformation and they do not slip at the contact surface.

The first method of analysis uses the following assumptions. 1. In both layers the length of a linear element normal to the coordinate surface remains unchanged after the deformation ($\varepsilon_{\gamma\gamma}=0$). 2. The shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ are approximately the same as in the case where the normals for the whole shells (both layers) remain undeformed.

In the second method the first assumption is not used.

The author solves the problem using both methods, compares both solutions and obtains the correction term arising from the first assumption. He refers very often to his previous works, and without these references [Prikl. Mat. Meh. **11** (1947), 527–532; **18** (1954), 303–312; Akad. Nauk Armyan. SSR. Dokl. **8** (1948), no. 5; Akad. Nauk Armyan. SSR. Izv. Fiz. Mat. Estest. Nauk **4** (1951), no. 5; **6** (1953), no. 3; MR **9**, 397; **15**, 1003], it is impossible to follow his derivations. *T. Leser* (Aberdeen, Md.).

Nowiński, Jerzy. Application of some familiar approximate methods to problems concerning orthotropic plates with large deflections. Rozprawy Inż. **5** (1957), 331–354. (Polish. Russian and English summaries)

In the case of large deflections the non-linear terms in the system of equations for plates cannot be neglected. For isotropic plates, approximate methods for solving this non-linear system of equations are well known. Applying some of these well known methods to the system of non-linear equations controlling orthotropic plates, the author solves the system for several different cases. He begins his paper by stating the Rostovcev system of equations for orthotropic plates, both in rectangular and polar coordinates, as given by S. G. Lechnicki, [Anisotropic plates, OGIZ, Moscow-Leningrad, 1947; MR **10**, 415]. He then solves the following cases. 1. a circular cylindrically orthotropic plate bent into a portion of a spherical surface. 2. a freely supported circular cylindrically orthotropic plate with deflections equal to those computed by linear theory (applicable for small deflections) for uniformly distributed load. As expected, the loads computed by the non-linear theory are not uniform. 3. a circular cylindrically orthotropic plate on a hinged support, uniformly loaded. 4. a rectilinearly orthotropic elliptic plate on a hinged support, uniformly loaded. 5. a rectangular orthotropic plate uniformly loaded, with supports which prevent horizontal displacements (edges fixed in the horizontal plane). In cases 3, 4, 5, the author applied the pseudo-energy method introduced by J. Prescott [Applied elasticity, Longmans-Green, London, 1924]. 6. a freely supported square orthotropic plate, uniformly loaded. 7. a square orthotropic plate supported on hinges and uniformly loaded. For the last two cases, the author applied the energy method introduced by A. and L. Föppl and described by S. Timoshenko [Theory of plates and shells, McGraw-Hill, New York, 1940]. *T. Leser* (Aberdeen, Md.).

Iwiński, Tadeusz; and Nowiński, Jerzy. The problem of large deflections of orthotropic plates. I. Arch. Mech. Stos. **9** (1957), 593–603. (Polish and Russian summaries)

A generalization is presented of the procedure of H. M. Berger to the case of orthotropic plates. The method neglects the so-called second strain invariant of the middle plane strain in the expression for unit strain energy. Results are obtained for a circular plate uniformly loaded for a particular value of orthotropy. The results are compared with an approximate solution obtained by means of a pseudo-energy equation. Fair agreement between the two results is found. The numerical results are given for deflections up to about 1.5 plate thicknesses. *S. Levy* (Washington, D.C.).

Johnson, Millard W.; and Reissner, Eric. On transverse vibrations of shallow spherical shells. Quart. Appl. Math. **15** (1958), 367–380.

The vibrations, without axial symmetry, of a spherical shell segment (or cap) with free edges are considered. The dependence on the number of nodal circles and nodal radii is studied in detail. An important parameter in the results is a reference frequency p_∞ which depends on the geometric and mechanical properties of the shell.

It is found for the cases of symmetric and one nodal diameter-vibrations that p_∞ is the lowest possible frequency of free vibration. Furthermore, the frequencies can be obtained by a transformation of the solutions to Kirchhoff's frequency equation for vibrations of circular plates with free edge, with zero or one nodal diameter. When the bending stiffness approaches zero or the ratio of cap height to wall thickness becomes infinite the limiting frequency is p_∞ for one or more nodal circles. In this sense p_∞ can be considered to be the membrane vibration frequency of the shell.

With two or more nodal radii, frequencies less than p_∞ are possible. The solutions of the frequency equation for two nodal radii are carried out explicitly. Again part of the results can be found from the flat plate equation but there is a difference depending on whether there are no nodal circles or one or more. In the first case, shell curvature is significant whereas in the second it is not. The limiting cases of membrane and inextensional vibrations also depend on whether there are nodal circles or not.

Energy calculations illustrate this difference in behavior. As the ratio of cap height to wall thickness varies, the strain energy shifts from bending to stretching for one or more nodal circles whereas it shifts from bending to stretching and then back to bending for no nodal circles. *G. H. Handelman* (Troy, N.Y.).

Kato, Tosio; Fujita, Hiroshi; Nakata, Yoshimoto; and Newman, Morris. Estimation of the frequencies of thin elastic plates with free edges. J. Res. Nat. Bur. Standards **59** (1957), 169–186.

Kato's method for obtaining upper and lower bounds for the eigenvalues of an operator of the form T^*T [J. Phys. Soc. Japan **4** (1949), 334–339; MR **12**, 447] is applied to the vibrations of a free plate. Numerical bounds

accurate to one part in 1000 are obtained for the lowest positive eigenvalue of the square plate. The method used is the same as that used by Nakata and Fujita [ibid. **10** (1955), 823–824; MR **17**, 804], but more arbitrary functions are introduced and the bounds obtained are more accurate.

The last section discusses plates with other boundary conditions. In connection with clamped plates, the present method is compared to that of A. Weinstein and N. Aronszajn. It is noted that both methods require knowledge about an auxiliary problem, and the conclusion is drawn that the range of applicability of the two methods is about the same.

This is an inaccurate conclusion. The Weinstein-Aronszajn method requires a solvable eigenvalue problem having lower eigenvalues than the given problem. In the case of any clamped plate, such a problem can always be found from the monotonicity properties. The Kato method, on the other hand, requires knowledge of a number β which is simultaneously an upper bound for the eigenvalue in question and a lower bound for the next eigenvalue. One must also have a number α which is a lower bound for the eigenvalue in question and, if this eigenvalue is not the lowest, an upper bound for the next lower one. Furthermore, one must have unit vectors u, v such that $\alpha<(Tu, v)<\beta$. It is difficult to see how these quantities can be obtained in a general case. All the symmetry of the square plate had to be mobilized to give the results in this paper.

This is not to deny, however, that in those special cases where the Kato method is applicable, it gives elegant, useful, and, as the present paper shows, accurate results.

H. F. Weinberger (College Park, Md.).

Mandel, Jean. Sur les vibrations des corps élastiques. C. R. Acad. Sci. Paris **245** (1957), 2004–2006.

In this short note, the author describes the forced vibrations of an elastic body, by means of a set of integral equations involving a Green's or "action" tensor as kernel. The Fredholm theory is then employed to obtain a series expansion for the displacements. *W. E. Boyce.*

★ **Malvern, L. E. Effect of damping on vibration frequencies of simple systems.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 195–205. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

The author treats the effect of damping on the vibrations of a one-mass system, where the damping is applied at some point on the spring rather than to the mass itself. It is shown that viscous damping applied on the spring may raise the peak response frequency of the system, in contrast to the lowered frequency when the damping is applied directly to the mass. For light damping, the transition occurs when approximately one-third of the spring compliance is between the mass and the damper. Furthermore, Coulomb friction applied anywhere except at the mass reduces the time between successive displacement maxima in free vibration below the period of the undamped system. Qualitative experimental verification of these predictions was obtained with relatively simple tests on a torsional pendulum. *W. E. Boyce.*

Payne, L. E.; and Weinberger, H. F. Lower bounds for vibration frequencies of elastically supported membranes and plates. J. Soc. Indust. Appl. Math. **5** (1957), 171–182.

Consider the two-dimensional problem defined by $\Delta u+\lambda u=0$ on a region R, and $\partial u/\partial n+ku=0$ on its boundary. The intersection of R with a line parallel to the y-axis will consist of a finite number of intervals. Suppose the maximum length of all such intervals is L_x. Define L_y in a similar way. Then it is shown that the smallest eigenvalue $\lambda_1(k)\geqq\mu_1(k_1, L_x)+\mu_1(k_2, L_y)$. Here $\mu_1(k_1, L_x)$ is the relatively easily found smallest eigenvalue of the one-dimensional problem $v''+\mu v=0$, $-v'(0)+k_1v(0)=v'(L_x)+k_1v(L_x)=0$ (and similarly $\mu_1(k_2, L_y)$), and $k_1|\partial x/\partial n|+k_2|\partial y/\partial n|\leq k$ on the boundary of R. It is also shown that $\lambda_1(k)$ is greater than the corresponding eigenvalue for the circle which circumscribes R. Analogous results are given for the three-dimensional problem. A lower bound for $\lambda_2(k)$ is given under certain symmetry restrictions on R.

Elastically supported plates are also considered. A lower bound for the smallest eigenvalue is given, and lower bounds are given for all the eigenvalues in terms of those for membranes as well as one-dimensional problems.

T. E. Hull (Pasadena, Calif.).

Cohen, Hirsh; and Handelman, George. On the vibration of a circular membrane with added mass. J. Acoust. Soc. Amer. **29** (1957), 229–233.

Natural frequencies of membrane in form of circular annulus supported at outer radius and attached to concentric rigid central mass are studied. Effect of density ratio between central mass and membrane, and of radius ratio of annulus is considered for radially symmetrical motion. Added mass may increase natural frequencies if associated increase of inner radius increases the stiffness sufficiently. Variational theorems and asymptotic expansions for higher modes are used to determine influence of various parameters. *E. H. Lee* (Providence, R.I.).

★ **Crandall, Stephen H. The Timoshenko beam on an elastic foundation.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 146–159. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

Pavlov, I. G. The fundamental equations for combined stability and strength of thin-walled open section bars. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **3** (1957), 306–316. (Ukrainian. Russian and English summaries)

Combined bending and compression are considered. From the solution of particular problems it appears that failure may occur long before the critical load (as ordinarily calculated) is reached. *R. C. T. Smith.*

★ **Nalwszkiewicz, J. Energy levels in dynamics of elastic systems.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 111–126. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

This paper is identical, except for absence of numerical results, with part one of the author's previous paper of the same title [Arch. Mech. Stos. **8** (1956), 471–506; MR **19**, 83].

W. E. Boyce (Troy, N.Y.).

★ **Taylor, C. E. Elastic stability of conical shells loaded by uniform external pressure.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 86–99. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

Die Kegelschalen spielen heute in vielen Gebieten der Technik, so vor allem beim Bau von Unterseebooten und

Geschossen, sowie bei der Konstruktion von Druckbehältern, eine große Rolle. In allen diesen Fällen kann man die Belastung näherungsweise als gleichförmigen Normaldruck ansehen. Es interessiert nun besonders die Beullast dieser Schalen; zahlreiche Versuche in dieser Richtung zeigten wenig befriedigende Übereinstimmung mit den theoretisch berechneten Werten.

Der Verfasser behandelt das erwähnte Problem nach dem bekannten Energiekriterium. Den Ausdruck für die Formänderungsenergie leitet er auf Grund der allgemein üblichen Voraussetzungen der linearen Schalentheorie ab. In den Energieausdrücken werden auch quadratische Terme berücksichtigt, so daß man darauf ohne Weiteres auch eine Theorie für große Formänderungen aufbauen kann. Selbst wenn man die quadratischen Glieder fortläßt, sind die Ausdrücke noch außerordentlich lang und unübersichtlich. Eine noch weitergehende Vereinfachung, wie sie von anderen Autoren vorgeschlagen wurde, hält der Verfasser aber im Hinblick auf die dann zu Tage tretenden Diskrepansen zwischen Rechnung und Experiment für nicht zulässig. Mit Hilfe des wohlbekannten Verfahrens von Rayleigh-Ritz wird schließlich eine obere Grenze für die Beulspannung im Falle infinitesimaler Verformungen bestimmt. *W. Zerna* (Hannover).

Eimer, Czesław. The stability of prestressed elements by means of binding. Rozprawy Inż. **4** (1956), 543–564. (Polish. Russian and English summaries)

The equation is established of the deformed shape of an elastic thin-walled circular tube of infinite length subject to a radial prestress exerted by elastic wires uniformly distributed over its length. Solutions are obtained in the form of elliptic integrals from which the buckling hoop-force is derived. *A. M. Freudenthal* (New York, N.Y.).

Hu, Hai-chang. On reciprocal theorems in the dynamics of elastic bodies and some applications. Sci. Sinica **7** (1958), 137–150.

The author considers the dynamic behavior of a linear, non-isotropic, non-homogeneous elastic body. Applying Laplace transforms with respect to time, the transformed problem is seen to be similar to a static problem for which the reciprocal theorem of Betti is known. For bodies which are initially at rest, have an immovable supporting surface, and are loaded by variable loads keeping their relative magnitudes invariable, there is a reciprocal relation between displacements and forces at any instant. A similar result is found for bodies with linear damping. Free oscillations and specific examples are considered. (It should be noted that the basic theorems are stated for distributed loads although concentrated loads are considered in the applications.) *G. H. Handelman.*

Goldsmith, Werner. An elongating string under the action of a transverse force. J. Appl. Mech. **24** (1957), 609–616.

In order to investigate certain rocket problems, the author studies the transverse motion of an elongating flexible string, fixed at one end, stretched by a constant tensile force acting in the direction of increasing length and driven transversely by a known force. The technique of solution is applicable to the subsonic case only (longitudinal velocity of the free end less than the sound velocity in the string), and clearly indicates the different character of the problem for the supersonic case.

The mathematical problem is reduced to the solution of a wave equation with one moving boundary condition. The method of attack is based on the method of characteristics and the Riemann-Volterra technique. An integral expression for the displacement is found in terms of the unknown transverse motion of the moving end. This function is then found by an appropriate limiting process followed by a judicious application of a resulting recursion formula. The cases in which the end of the string travels at constant velocity and also at constant acceleration are worked out.

A numerical example is given for a harmonic driving force. For the case of constant end velocity, the solutions are bounded and no resonance takes place. It is also pointed out that, in the supersonic case, the method of characteristics shows that the solution is governed by the initial conditions rather than the boundary conditions at the fixed end. This mirrors the physical situation, in that disturbances at the free end will not reach the fixed end and be reflected back. *G. H. Handelman* (Troy, N.Y.).

Ringleb, F. O. Motion and stress of an elastic cable due to impact. J. Appl. Mech. **24** (1957), 417–425.

This paper deals with an initially straight elastic cable, one point of which suddenly moves with constant velocity V_0, at an angle β to the axis of the cable. The equations of motion are derived, and a formula is obtained relating the impact velocity and the impact stress to the angle β and the initial tension in the cable. An approximation to this formula, adequate for steel cables, is represented graphically. For the case of perpendicular impact with zero initial stress, comparisons with experimental data yield satisfactory results. The distribution of input energy into kinetic and potential energy of the various parts of the cable is also discussed.

W. E. Boyce (Troy, N.Y.).

Roseau, Maurice. Sur un théorème d'unicité applicable à certains problèmes de diffraction d'ondes élastiques. C. R. Acad. Sci. Paris **245** (1957), 1780–1782.

The author announces a uniqueness theorem for the propagation of elastic waves in an infinite elastic and homogeneous medium, subject to certain conditions.

A. E. Heins (Pittsburgh, Pa.).

Gorshkov, N. F. On pulse propagation in a dissipative elastic medium. Akust. Ž. **3** (1957), 154–162. (Russian)

The author considers a plane pressure wave, propagated in a semi-infinite medium, which is caused by a pressure impulse at the boundary plane $x=0$. The disturbing pressure at the boundary is expressed as the unit function $f(0, t)=1$ when $t>0$, and $f(0, t)=0$ when $t<0$.

The problem posed by the author was to find $f(x, t)$ when the damping factor is of the form $\exp(-\alpha\omega^n)$, where ω is the circular frequency, α is the damping coefficient, and n is a non-zero positive integer. Using a Fourier integral and assuming $n=2$, the author derives $f(x, t)$, which turns out to be an expression containing error which function of t and α, and which is independent of x. The solution shows that away from the boundary the damping inclines, or generally speaking deforms, the initially rectangular front of the impulse, the measure of the inclination being the time derivative of f. Similarly, the author finds $f(x, t)$ when $f(0, t)$ is a rectangular pulse lasting from $t=0$ to $t=T$ and for $n=1$. The solution which is in terms of inverse tangents also shows pronounced deformation of the pulse front.

The author next investigates the determination of the

coefficient of damping α when the deformation of the pulse front is measured. He describes an experimental setup for measuring pulse forms propagated in the air, and presents some computed values of α. *T. Leser.*

Husein-Zade, M. I. Impact at an infinite plate lying on elastic liquid semispace. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 523–526. (Russian)

The author investigates the response to an impulsive force (impact) of an infinite elastic plate lying on a liquid semispace. When subjected to an impact, the liquid behaves like an elastic body whose second Lamé constant μ is zero. The author uses polar coordinates; the pole coincides with the point of application of the impulsive force. He writes down a system of equations consisting of the equation for transverse vibrations of a plate and the equation for the displacement potential function $\phi(r, z, t)$, with appropriate boundary conditions for each equation. The system is solved by the following sequence of operations: 1) the separation of variables in the first equation and the incomplete separation of variables in the second equation; 2) the use of Laplace transforms of the resulting equations; and 3) the attainment of the solution from the inverse transforms; finding these is essentially the main task. *T. Leser* (Aberdeen, Md.).

★**Eirich, F. R. (Editor). Rheology: theory and applications. Vol. 1.** Academic Press, Inc., New York, 1956. xiv+761 pp.

This is the first of three volumes which are intended to provide a comprehensive survey of rheology and to cover both its theoretical and experimental aspects.

This first volume is concerned with the more fundamental parts of the subject. After a general introduction by F. R. Eirich there are sixteen articles by different authors, dealing with various aspects of deformation and flow in a concise, authoritative and thorough manner.

There are a number of ways in which rheological problems may be approached, and the articles in this book are properly representative of these several approaches.

There are, first, general phenomenological treatments, where the consequences of such factors as the geometrical limitations on the deformation of a continuous medium and the existence of a strain energy function are considered. Three of the articles which fall into this category are "Phenomenological macrorheology" by M. Reiner, "Large elastic deformations" by R. S. Rivlin, and "Non-Newtonian flow of liquids and solids" by J. G. Oldroyd.

Secondly, there is the more specialised phenomenological approach, where specific laws of deformation are assumed on the basis of the observed behavior of some real materials. Articles which discuss the consequences of such laws are "Finite plastic deformation" by William Prager, "Stress-strain relations in the plastic range of metals—experiments and basic concepts" by D. C. Drucker, "Dynamics of viscoelastic behavior" by Turner Alfrey, Jr. and E. F. Gurnee, and "Some rheological properties under high pressure" by R. B. Dow.

Finally, there is the microscopic or molecular approach, where the mechanical behavior of the bulk material is shown to be a consequence of its molecular or microcrystalline structure. The three articles which exemplify this approach for metals are "Mechanical properties and imperfections in crystals" by G. J. Dienes, "Dislocations in crystal lattices" by J. M. Burgers and W. G. Burgers, and "Mechanical properties of metals" by J. Fleeman and G. J. Dienes, whilst for liquids and suspensions the microscopic approach is given in "Theories of viscosity" by A. Bondi, "The statistical mechanical theory of irreversible processes in solutions of macromolecules" by J. Riseman and J. G. Kirkwood, "Viscosity relationships for polymers in bulk and in concentrated solution" by T. G. Fox, Serge Gratch, and S. Loshaek, and "The viscosity of colloidal suspensions and macromolecular solutions" by H. L. Frisch and Robert Simha.

In addition to the fourteen articles which have been mentioned above in somewhat arbitrary groupings, there are two more specialised articles in this first volume. The first of these is a full discussion of "Streaming and stress birefringence" by A. Peterlin, and the second "Acoustics and the liquid state" by R. B. Lindsay, which includes an account of how ultrasonic investigations help in the study of liquid structure.

Although there is a certain amount of overlap between these separate articles, and it is inevitable that some readers will find their particular rheological interests treated rather briefly, the book gives a well-balanced picture of the present state of the subject, and will undoubtedly become a standard reference work.

This volume is excellently produced and well indexed, and the everpresent bugbear of differing nomenclatures is to a large extent overcome by separate lists of symbols at the end of each article. *H. Kolsky.*

Miyake, Akira. General theory of relaxation phenomena. Nat. Sci. Rep. Lib. Arts Fac. Shizuoka Univ. no. **9** (1956), 3–8.

The general equations of linear viscoelasticity are discussed in terms of thermodynamics of irreversible processes. Subsequently, the statistical foundation of the thermodynamical formulation is established. *B. Gross.*

Miyake, Akira. Notes on the general theory of relaxation. Nat. Sci. Rep. Lib. Arts Fac. Shizuoka Univ. no. **10** (1957), 1–5.

In continuation of the paper reviewed above, the general equations of linear viscoelasticity are written in matrix form. A dissipation function is established from which these equations can be derived. As an application, mechanical relaxation time spectra for solid polymers are discussed. *B. Gross* (Rio de Janeiro).

Nakada, Osamu. Theory of visco-elasticity of amorphous polymers. J. Phys. Soc. Japan **10** (1955), 804–813.

Intra-chain relaxation in amorphous polymers is discussed in terms of a "random spring model". No distinction is made between permanent chemical crosslinks and temporary links caused by chain entanglement because it is supposed that the period of the applied force is small compared with the mean life time of the temporary links. The diffusion equation of the problem is solved under the assumption of a suitable potential function. The resulting spectrum of relaxation times has the form of the wedge spectrum which has been observed for polyisobutylene. *B, Gross* (Rio de Janeiro).

Saitô, Nobuhiko; and Kato, Tomokazu. On the visco-elasticity and complex dielectric constant in the presence of an electric field and a shearing laminar flow in solution of macromolecules. J. Phys. Soc. Japan **12** (1957), 1393–1402.

The paper discusses the interaction effects between an

applied electric field and streaming laminar flow in solution of polar macromolecules. Such effects are the influence of the electric field on the intrinsic viscosity and the influence of the velocity gradient of flow on the dielectric properties of the system. Treatment is based on a model in which polar macromolecules are considered as spheroidal rigid particles with permanent dipoles. It is shown that the intrinsic viscosity increases proportionally to the square of the applied electric field and the dielectric constant proportionally to the square of the velocity gradient. *B. Gross* (Rio de Janeiro).

Yamamoto, Misazo. The visco-elastic properties of network structure. I. General formalism. J. Phys. Soc. Japan **11** (1956), 413–421.

Behavior of amorphous high polymers is discussed in terms of a modified rubber-like network model in which the junctions connecting the polymer chains break up and reform continually. The probability of breakage per unit time is a function of the number of segments and of the end-to-end distance of the chain. The rates of change of macroscopic conditions, for instance the rate of deformation, are supposed to be very much smaller than those of the micro-Brownian movement of the polymer segments. Therefore equilibrium statistical mechanics can be applied. Stress-strain relations result which contain the measurement time as a parameter and therefore are valid at each "instant" of macroscopic observation.

B. Gross (Rio de Janeiro).

Yamamoto, Misazo. The visco-elastic properties of network structure. II. Structural viscosity. J. Phys. Soc. Japan **12** (1957), 1148–1158.

Viscosity behavior of concentrated polymer solutions is treated by the model of a weak Gaussian network. If the chain-breakage coefficient, i.e. the probability of breakage of a chain in the network, is independent of the elongation of the chain, such systems show Newtonian flow. Proper choice of the chain breakage coefficient, for example that of the square-well type, yields a characteristic rate dependence of viscosity; the larger the rate of shearing flow, the smaller is the value of the viscosity coefficient. The tensile viscosity is also considered.

B. Gross (Rio de Janeiro).

Paslay, P. R. Calendering of a viscoelastic material. J. Appl. Mech. **24** (1957), 602–608.

Calendering of a Maxwell material, with viscosity also varying in a restricted manner, is considered in this paper. Once the stress-strain rate-time assumption is established as reasonable, approximations much like those in other studies of calendering are made. A numerical procedure is outlined for solving the resulting set of equations, and several such calculations are carried out in special cases.

R. E. Gaskell (Seattle, Wash.).

Segawa, Wataru. Stress-strain relations for viscoelastic large deformation. J. Phys. Soc. Japan **12** (1957), 996–998.

The finite strain incompressible stress-strain relations of rubber elasticity are generalized for viscoelasticity by replacing the elastic constant by a Maxwell and a Kelvin differential operator, while retaining finite strain components. It is not obvious to the reviewer that these would satisfy the required invariance relations when time and invariance under angular velocity are introduced. The laws are applied to simple particular problems of homogeneous axial strain. *E. H. Lee* (Providence, R.I.).

Radok, J. R. M. Visco-elastic stress analysis. Quart. Appl. Math. **15** (1957), 198–202.

The author has modified Lee's method [same Quart. **13** (1955), 183–190; MR **16**, 1073] for solving certain quasi-static problems in linear viscoelasticity. However, the reviewer feels that the author has not proved that his treatment is valid in the cases for which Lee's method is not applicable. *D. R. Bland* (Manchester).

Biot, M. A. Folding instability of a layered viscoelastic medium under compression. Proc. Roy. Soc. London. Ser. A. **242** (1957), 444–454.

The author treats the stability of a viscoelastic layer of constant thickness, sandwiched in an infinitely extended medium of another viscoelastic material and subject to uniform pressure in its plane. The elastic solutions, both for the stability of a thin plate and for the deflection of a semi-infinite plane under normal surface loading, can be adopted for viscoelastic materials if the elastic constants are replaced by the appropriate time operators. The combination of the two solutions gives an equation connecting the pressure and the wavelength of the distortion, as for an elastic plate in an elastic medium, and another parameter representing the time rate of growth of the amplitude of the distortion. For a given pressure, the dominant wavelength is obtained by maximising this parameter. No buckling occurs for pressures less than that for which the maximum value of the parameter is zero, and instantaneous buckling occurs for pressures greater than that for which the maximum first becomes infinite. Special cases, i.e., specific types of viscoelastic material in layer and medium, are discussed in more detail.

D. R. Bland (Manchester).

Petrova, S. G. On the first boundary problem of the non-linear theory of elasticity. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 41–44. (Russian)

The equations of equilibrium are $P\mathbf{u}=0$, where $\mathbf{u}$ is the (vector) displacement, and $P\mathbf{u}=\operatorname{div} T+K$, where K is the given body force per unit volume and T the stress tensor. A non-linear stress-strain relation is assumed,

$$T=\psi(\Gamma^2)E+\{k-\tfrac{1}{3}\psi(\Gamma^2)\}\varepsilon I,$$

where $E=(\varepsilon_{ij})$ is the stress tensor, I the unit tensor, $\varepsilon=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$ and $\psi(\Gamma^2)$ the plasticity function, Γ being the deformation.

Successive approximations may be obtained from $P_{(\mathbf{u}_n)'}\mathbf{u}_{n+1}=P_{(\mathbf{u}_n)'}\mathbf{u}_n-P\mathbf{u}_n$ (Newton's method), or from $P_{(0)'}\mathbf{u}_{n+1}=P_{(0)'}\mathbf{u}_n-P\mathbf{u}_n$ (modified Newton's method), where $P_{(\mathbf{u}_n')}$ is the Fréchet differential of the operator P.

Using the modified Newton's method, bounds are obtained for $\|\mathbf{u}\|$ and $\|\mathbf{u}-\mathbf{u}_n\|$ under suitable assumptions on $\psi(\Gamma^2)$ and K. Various norms are used, the most complicated involving all derivatives from zeroth to fourth order. *R. C. T. Smith* (Armidale).

Storchi, Edoardo. Soluzioni ad un parametro del problema plastico della deformazione piana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 286–293.

For the plastic region of a non-hardening von Mises or Tresca material in plane strain, the author finds solutions of the form $p=p(\phi)$, where $p=-\frac{1}{2}(\sigma_{xx}+\sigma_{yy})$ and $\phi=\frac{1}{2}\arctan\frac{\sigma_{yy}-\sigma_{xx}}{2\sigma_{xy}}$. Either $p+2k\phi$ or $p-2k\phi$ must be constant and hence, by Hencky's theorem, one set of slip

lines must be straight. The method is generalised to yield criteria of the form $F(\sigma_{zz}, \sigma_{yy}, \sigma_{xy})=0$. *D. R. Bland.*

Shaffer, Bernard W.; and House, Raymond N., Jr. Displacements in a wide curved bar subjected to pure elastic-plastic bending. J. Appl. Mech. **24** (1957), 447–452.

An elasto-plastic non-hardening wide curved bar is subject to a bending moment in the sense that tends to increase the curvature of the bar. If the moment is steadily increased from zero, the deformation at first is entirely elastic, then a plastic region forms around the concave surface of the bar, next a plastic region forms around the convex surface and, finally, these two regions meet in the interior. The stresses and displacements are found in all cases. The displacements are small enough to justify referring the boundary conditions to the original shape of the bar, provided that the bending moment is less than 95% of its ultimate value. *D. R. Bland.*

Hill, R. Stability of rigid-plastic solids. J. Mech. Phys. Solids **6** (1957), 1–8.

The sufficient condition for stability that the work of deformation exceeds the applied work for all small displacements is exploited. In a rigid-plastic body, these will in general correspond to finite changes in stress, but the special modes of displacement associated with the stress by the flow condition are more crucial. The test functional for stability is the same as that obtained previously for uniqueness but the admissible displacement functions form a wider class for uniqueness. This gives differing critical parameters, and in a particular example of a strut it is shown that the critical hardening rate for stability is one fifth of that for uniqueness. Bifurcation of solutions can occur before instability. *E. H. Lee.*

Ross, E. W., Jr. On effect of boundary and loading conditions in the limit analysis of plastic structures. J. Appl. Mech. **24** (1957), 314–315.

Theorems concerning the change in the limit load for certain geometrical and mechanical changes in an assemblage are given. They include increase in size by expanding a stress free boundary, and by expanding a motion free boundary. Changing the width of a smooth indenter, and replacing a smooth indenter by a distribution of normal pressure are also considered. Proofs follow directly from limit analysis theorems. *E. H. Lee.*

Hult, J. A. H. Elastic-plastic torsion of sharply notched bars. J. Mech. Phys. Solids **6** (1957), 79–82.

From analogy with the known situation in pure shear, the stress field in elastic-plastic torsion including the determination of the elastic-plastic boundary is presented for a sharp wedge-shaped notch. The result is in agreement with Trefftz' solution for an *L*-shaped beam. *E. H. Lee* (Providence, R.I.).

Boyce, William E. The plastic bending of an eccentrically loaded column. J. Aero. Sci. **24** (1957), 332–338, 362.

The beam-column considered is assumed to be initially straight and subject to eccentric axial loads which increase slowly from zero. The column is of the sandwich type consisting of two identical flanges carrying all direct stresses and a central layer supporting only shear stresses. The material is rigid work-hardening and hence only the nonconservative effects are emphasized. The stress-strain law is of the type discussed by Prager [Proc. Inst. Mech. Engrs. **169** (1955), 41–57; MR **17**, 558]. The special case of a column which is rigid save for a centrally deformable element, as well as the general column is studied. Deflection versus load behavior is similar in both cases. For small eccentricities, the total displacement will be small until both flanges become plastic. After this, further appreciable displacements will not take place until a considerably higher load is reached. These results show that care must be taken in defining "buckling loads" for such structures. *G. H. Handelman.*

Hill, R. On the problem of uniqueness in the theory of a rigid-plastic solid. IV. J. Mech. Phys. Solids **5** (1957), 302–307.

The author continues his study of uniqueness questions in plasticity [same J. **4** (1956), 247–255; **5**, 1–8 (1956), 153–161 (1957); MR **18**, 83, 690; **19**, 597] by considering a rigid-plastic body in which the traction and traction-rate are given on part of the surface and the velocity on the remainder. By modifying a sufficient condition for uniqueness, previously given, an extremum principle is stated and proved which takes into account changes in geometry. The uniqueness criterion did not depend on the applied traction rate, whereas the extremum principle explicitly brings in this quantity. Various remarks on the implications and limitations of the result are given. Alternative expressions for the functional are derived and the particular cases of uniaxial tension and plane strain are carried out in detail. *G. H. Handelman.*

Shield, R. T. The application of limit analysis to the determination of the strength of butt joints. Quart. Appl. Math. **15** (1957), 139–147.

Plastic limit analysis is used to obtain upper and lower bounds to the yield point of a thin rigid-plastic lamina compressed between parallel plates to which it adheres. The Tresca yield condition and flow rule are assumed. The author unprecisely states that, if geometry changes are disregarded, an elastic-plastic lamina would be deformed progressively at the yield point without further increase in load. It ought to be remarked, however, that the rigid-plastic yield point load is normally only approached asymptotically in an elastic-plastic solid, and is not actually attainable. Such a gradual approach arises through the influence of elastic compressibility in the plastic zone, and also through the existence of an elastic zone which diminishes but never entirely vanishes. *R. Hill* (Nottingham).

Thomas, T. Y. The Lüders band problem. J. Math. Mech. **7** (1958), 17–27.

Despite the title, the paper deals only with a small part of the Lüders band problem, as it is understood by metallurgists. The author discusses the occurrence of regular oblique bands of yielded material in tension tests on thin strips of certain metals which have distinct upper and lower yield points. The first part of the paper is a complicated derivation of the well-known result that, if the bands can be formally treated as incipient localized necks in an ideal rigid-plastic solid, they should be inclined at $\tan^{-1}\sqrt{2}$ to the tension axis. In the second part the same result is reached when allowance is made for the elastic component of strain in the basic stress-strain relations, and when it is assumed that the load remains constant during the formation of the neck. *R. Hill* (Nottingham).

Dorn, W. S.; and Greenberg, H. J. Linear programming and plastic limit analysis of structures. Quart. Appl. Math. **15** (1957), 155–167.

The paper presents a reduction of problems of limit analysis to three basic types of linear programming problems, and a review of appropriate methods of solution. Simple examples illustrate the procedures. It is emphasized that the value of linear programming solutions lies in the analysis of problems in many variables which arise in highly indeterminate structures, and which may now be analysed by systematic routine procedures for high-speed digital computers. Available codes for various computers are listed. *W. T. Koiter* (Delft).

Haythornthwaite, R. M.; and Shield, R. T. A note on the deformable region in a rigid-plastic structure. J. Mech. Phys. Solids **6** (1958), 127–131.

J. F. W. Bishop, A. P. Green and R. Hill [same J. **4** (1956), 256–258; MR **18**, 83] have presented a procedure for finding the greatest extent of the deformable region in a plastically-yielding body. This procedure applies to work-hardening plastic-rigid material; there the yield function and plastic potential coincide. This paper gives an extension to structures such as beams, plates and shells, and now approximate plasticity theories concerning generalized stresses and strain-rates are involved. As an example, a flat, clamped, arbitrarily-shaped plate loaded by a transverse concentrated force is considered. Tresca's yield condition is assumed. The deformable region is proved to be a circle, with its centre at the point of application of load, and passing through the point on the plate's edge nearest to the load. *H. G. Hopkins.*

Teplitsky, E. I. The plane problem of contact between a punch and the deformed semi-plane in elasticity and limit equilibrium theory. Akad. Nau Ukraïn. RSR. Prikl. Meh. **3** (1957), 277–288. (Ukrainian. Russian and English summaries)

This paper discusses the theoretical soil-plasticity problem of the title. In the analysis, Coulomb's yield condition is assumed, and attention appears to be confined mainly to the stress field.

"The problem of contact between a punch and the semi-plane being deformed is examined. A system of three integral equations which, when solved, determine the border and stress distribution in the zone of elasticity, is obtained from the condition of equal tangential and normal stresses on both sides of the border between the zone of elasticity and that of limit equilibrium. An approximate method of solution, where the above-mentioned conditions are satisfied at a finite number of points on the border, is suggested. In this case, the problem is reduced to the solution of a system of non-linear algebraic equations. A criterion for selecting the appropriate number of points is provided. The dependence $\Delta\alpha=f(l)$ was obtained, where $\Delta\alpha$ is the difference of displacements derived with and without taking into account the zones of limit equilibrium on the punch edges; l is the percentage of the punch area under which the ground is in a state of limit equilibrium". [From author's summary.] *H. G. Hopkins.*

Liberman, Yu. M. On the type of the equations in the theory of plasticity. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 32–34. (Russian)

The mathematical type of the stress equations is discussed for a general, ideal-plastic material (yield stress-states depending upon the sum of the principal stresses) under conditions of equilibrium and plane strain. A simple example is given. The results are mainly standard [see e.g., R. Hill, The mathematical theory of plasticity, Oxford, 1950, pp. 294 et seq.; MR **12**, 303].

H. G. Hopkins (Sevenoaks).

Ševčenko, K. N. Elastic-plastic deformation of a plane under the action of a concentrated force. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 473–474. (Russian)

An infinite medium is subjected to a concentrated line force, under conditions of plane strain. The classical solution of elasticity theory is given, and the paper then discusses the elastic-plastic solution on the basis of deformation theory, with specialization to incompressible material. The exact meaning to be given to the author's analysis is not clear to the reviewer, and the development of a correct approach to this problem would appear to him to be extremely difficult [cf. H. G. Hopkins, Proc. Cambridge Philos. Soc. **46** (1950), 164–181; MR **11**, 285].

H. G. Hopkins (Sevenoaks).

Weir, C. D. The creep of thick tubes under internal pressure. J. Appl. Mech. **24** (1957), 464–466.

The creep strain rate is assumed given in the form $\dot{E}_{ij}=\psi(\tau, T)S_{ij}$, where $\dot{E}_{ij}$ are the incompressible strain rate components, S_{ij} the stress deviator components, τ the octahedral shearing stress, T the temperature, and ψ an arbitrary function. Assuming rotationally symmetric plane strain, the author reduces to quadratures the solution to the title problem. His general solution is shown to reduce to various particular solutions previously obtained by other authors. *P. G. Hodge, Jr.*

Venkatraman, B.; and Hodge, P. G., Jr. Creep behaviour of circular plates. J. Mech. Phys. Solids **6** (1958), 163–176.

The creep bending of a circular plate under uniform lateral pressure load is investigated analytically by using the Tresca criterion (maximum shearing stress) and the associated flow rule. This criterion has been previously applied to the problem of the plastic bending of plates by Hopkins and Prager [same J. **2** (1953), 1–13; MR **15**, 270], and it has also been used as a basis for the analysis of creep in rotating disks by Wahl [J. Appl. Mech. **23** (1956), 231–238]. Theoretical predictions in Wahl's paper showed reasonably good agreement with the experimental results given there. In the spirit of this previous work, the following assumptions are made in the present paper: elastic deformations are negligibly small in comparison with deformations due to creep; steady creep conditions prevail, that is, creep takes place under essentially constant stress; and, finally, the creep rate is equal to the product of a function of the moment and a function of the time. Solutions for moments and creep deformations are presented for the cases where the moment function in the creep rate is chosen as a power function and as an exponential function. Also, solutions are given for both the simply supported and the fully clamped edge condition.

W. Nachbar (Palo Alto, Calif.).

Biot, M. A.; and Willis, D. G. The elastic coefficients of the theory of consolidation. J. Appl. Mech. **24** (1957), 594–601.

The theory of deformation of an anisotropic, porous, elastic solid containing a compressible fluid is characterized by 4 coefficients. Any combination of measurements which is sufficient to the properties of the system may be

used for the determination of these coefficients. The most convenient measurements are those of the shear modulus, jacketed and unjacketed compressibility, and the coefficient of fluid content. The stress-strain relations may be expressed in terms of stresses and strains produced during the various measurements to give 4 expressions relating the measured coefficients. The same method is easily extended to the cases of anisotropy. (From the author's summary.)

B. Gross (Rio de Janeiro).

Sharma, Brahmadev. Thermal stresses in a semi-infinite elastic solid due to periodic temperature distribution over a portion of its plane surface. Proc. Nat. Inst. Sci. India. Part. A. **23** (1957), 258–263.

In his previous paper [J. Appl. Mech. **23**, 527–531; MR **18**, 689], the author presented a method for solving problems of thermal stresses, and in this work he applies his method to the solution of a semi-infinite solid having a periodic supply of heat on a circular area in its plane boundary. The results, that is, the expressions for stresses and displacements, were obtained in a closed form. The author states that the problem arises in the case of a periodic supply of heat by a blow jet on a circular area. It would be interesting to learn how realistic is the author's model, in which the temperature is assumed to be $T=T_0 \sin(pt)$ inside the circle $r \leqq a$, and $T=0$ when $r>a$.

T. Leser (Aberdeen, Md.).

★ **McDowell, E. L. Thermal stresses in an infinite plate of arbitrary thickness.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 72–85. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

Steady temperature distributions are prescribed over both faces $z=0$ and $z=h$ of an isotropic elastic plate of infinite extent. From the classical equations of elasticity and heat conduction, formulas for steady-state thermal stresses and displacements are derived in terms of corresponding displacements in an elastic medium filling the half-space $z \geqq 0$. The problem for the half-space was treated earlier [E. Sternberg and E. L. McDowell, Quart. Appl. Math. **14** (1957), 381–398; MR **19**, 343]. The method involves Green's functions and superposition of solutions, giving displacements in the form of an infinite series whose terms represent displacements for half-space problems. An illustration gives surface stresses over one face when the temperature on a circular region of the face is a constant c while the rest of that face and the entire second face is maintained at temperature zero.

R. V. Churchill (Ann Arbor, Mich.).

McDowell, E. L.; and Sternberg, E. Axisymmetric thermal stresses in a spherical shell of arbitrary thickness. J. Appl. Mech. **24** (1957), 376–380.

An explicit series solution is given, exact within the classical theory of elasticity, for the steady-state thermal stresses and displacements induced in a spherical shell by an arbitrary axisymmetric distribution of surface temperatures. The corresponding solutions for a solid sphere and for a spherical cavity in an infinite medium are obtained as limiting cases. The convergence of the series solutions obtained is discussed. Numerical results are presented appropriate to a solid sphere if two hemispherical caps of its boundary are maintained at distinct uniform temperatures.

W. Nowacki (Bern).

Nowacki, Witold. The state of stress in an elastic space due to a source of heat varying with time in a harmonic manner. Rozprawy Inż. **5** (1957), 511–521. (Polish. Russian and English summaries)

The thermal stresses in the infinite elastic continuum due to a heat-source of harmonically varying strength are determined by solving the equation of the thermo-elastic displacement potential for the known temperature fields of a concentrated, a linear and a plane heat-source, neglecting the effects of inertia. *A. M. Freudenthal.*

Nowacki, Witold. A quasi-steady state three-dimensional thermo-elastic problem. Rozprawy Inż. **5** (1957), 499–509. (Polish. Russian and English summaries)

The thermal stresses in the elastic half-space, resulting from a heat-source within the free surface moving with constant velocity, are determined by removing the time-effect with the aid of a coordinate system moving with the source, using the well-known thermal solutions for the stationary heat source and integrating the equation of the thermo-elastic displacement potential for those solutions with the aid of Fourier integrals. The resulting non-zero stress components on the free surface are removed by super-position of a solution of the biharmonic equation with the aid of a Galerkin function. *A. M. Freudenthal.*

Nowacki, Witold. A steady-state three-dimensional thermo-elastic problem. Rozprawy Inż. **5** (1957), 489–497. (Polish. Russian and English summaries)

The thermal stresses within the elastic half-space subject to a given arbitrary temperature distribution over a finite region of the free surface are determined by using the Green's function for the heat-transfer problem of an infinitely small heat source within the surface, from which the temperature solution for the finite region is obtained by integration. Introducing this integral into the potential equation for th thermo-elastic displacement function and removing the resulting non-zero stress-components on the surface by superimposing a second solution fulfilling the free boundary condition derived from a Galerkin function, the solution of the general problem is obtained. *A. M. Freudenthal* (New York, N.Y.).

Green, A. E.; Radok, J. R. M.; and Rivlin, R. S. Thermo-elastic similarity laws. Quart. Appl. Math. **15** (1958), 381–393.

This paper is concerned with isotropic linearly elastic bodies in which Fourier's law of heat conduction is valid. It is assumed that there are no body forces and that the deformations are so slow that inertia may be neglected.

Assuming that the tractions and the temperature distribution on the surface are given, the authors derive the precise conditions under which the thermo-elastic stress in a body can be determined from that in a scale model. They give a detailed analysis of these conditions for plane strain and generalized plane stress, including the case in which a body in plane strain is compared with a model in generalized plane stress. *W. Noll.*

Ignaczak, Józef. Thermal stresses in a long cylinder heated in a discontinuous manner over the lateral surface. Arch. Mech. Stos. **10** (1958), 25–34. (Polish and Russian summaries)

The temperature stresses are found in a long solid circular cylinder under steady state conditions; the surface temperature is constant circumferentially and varies axially as Heaviside's step function. Integration of the

equations of thermo-elasticity is effected by introducing the thermoelastic potential of displacement.
E. H. Mansfield (Farnborough).

Seth, B. R. Finite thermal strain in spheres and circular cylinders. Arch. Mech. Stos. **9** (1957), 633–645. (Polish and Russian summaries)
Steady state temperature fields, spherically and axially symmetrical, in a thick-walled sphere and a thick-walled circular cylinder, respectively, are considered. Linear stress-strain law and finite strain tensor are assumed, by constant elastic coefficients and arbitrary variation of thermal unit volume expansion with temperature. The resulting differential equation, which is nonlinear, has been solved by using power series. Particular closed solutions for quadratic temperature law have been found.
J. Nowinski (Madison, Wis.).

See also: Geophysics: Fogagnolo Massaglia.

Structure of Matter

Morita, Tohru. Cell theory of classical liquid. Phase transition between gas, liquid and solid. I, II. J. Phys. Soc. Japan **12** (1957), 1195–1203.
This is a molecular theory of the cell-cluster type, which is apparently a condensation in English of an original in Japanese. The compression is severe and it is not possible without much labour to verify the calculations in detail, but the results reported (the critical constants and phase diagrams for liquid argon) are unusually good. *H. S. Green* (Suffolk).

Cohen, E. G. D. Application of the Kikuchi-Hijmans-De Boer method for order-disorder phenomena to the cell model of the liquid state. Physica **23** (1957), 801–815.
In the cell-cluster model for liquids a fundamental problem is to determine the number of ways of distributing geometrical figures on a lattice in such a way that no two of these overlap. Hijmans and de Boer [Physica **21** (1955), 471–484, 485–498, 499–516; MR **17**, 930] have previously used an approximation due to Kikuchi to solve this problem. This approximate solution is used in the paper under review to investigate the cell model of liquids, without restricting the number of molecules per cell or neglecting the correlation between configuration in groups of neighbouring cells. *H. S. Green.*

Van Kranendonk, J.; and Van Vleck, J. H. Spin waves. Rev. Mod. Phys. **30** (1958), 1–23.
This is an excellent elementary survey of the theory of spin waves, and its application to the magnetic properties of solids. Both equilibrium theory (for ferromagnets, antiferromagnets, and ferrimagnets) and non-equilibrium theory (relaxation and resonance absorption processes) are covered. *H. W. Lewis* (Madison, Wis.).

Vaĭnshteĭn, B. K. Application of convolution in the derivation of the formula for the temperature factor. Akad. Nauk SSSR. Kristallografiya **1** (1956), 137–138. (Russian)
In crystal structure analysis, the properties of convolution can be profitably used in the derivation of the well-known formula for the temperature factor of an atom subject to thermal vibration. *V. Vand.*

Sah, Chih-Tang; and Shockley, W. Electron-hole recombination statistics in semiconductors through flaws with many charge conditions. Phys. Rev. (2) **109** (1958), 1103–1115.
This paper is an extension of work by Shockley and Read [Phys. Rev. (2) **87** (1952), 835–842] and others, who studied the non-equilibrium statistics for holes and electrons through single-level flaws. The present article deals with the non-equilibrium but steady state statistics for holes and electrons in semi-conductors with a certain type of multiple-charge-condition flaw. In particular, the following situation is considered: a flaw with s electronic units of negative charge makes transitions to charge $s+1$ by hole emission at a rate of $e(p, s)$, or by electron capture at the rate $nc(n, s)$ and returns to charge s at rates $e(n, s+1)$ and $pc(p, s+1)$. Here n is the electron density in the conduction band and p the hole density in the valance band. The steady state population ratio is given, and expressions for the recombination rate based on steady state distribution are derived. The complexity of the problem is greatly reduced by the use of two diagrams, the R_N (charge distribution) and R_U (recombination rate) diagrams. *M. J. Moravcsik* (Livermore, Calif.).

Lonsdale, Kathleen. Thermal vibrations of atoms and molecules in crystals. Rev. Mod. Phys. **30** (1958), 168–170.
Reprinted from the proceedings of the international conference in crystal physics. See MR **19**, 84.

See also: Statistical Thermodynamics and Mechanics: Gurzhi. **Elasticity, Plasticity:** Saitô and Kato. **Quantum Mechanics:** Nozières and Pines.

Fluid Mechanics, Acoustics

★ **v. Krzywoblocki, M. Z.; and Whittenbury, C. G. On the possibility of the formulation of a quantum fluid dynamics theory.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 376–383. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.
The authors suggest a broad classification of numerous papers on the quantum theory of fluids (most of them published before 1955) and offer very brief comments on the papers cited. *C. D. Calsoyas* (Livermore, Calif.).

Maslen, Stephen H. Transverse velocities in fully developed flows. Quart. Appl. Math. **16** (1958), 173–175.
An incompressible viscous flow in x-direction is called fully developed if the driving force and the flow velocities are independent of x. Using the notion of the stream function, the author proves by means of an elementary mathematical technique that in such a flow the velocity components normal to the x-direction vanish identically. This fact is usually accepted by a physical intuition.
M. Z. v. Krzywoblocki (Urbana, Ill.).

★ **Strandhagen, A. G.; and Seikel, G. R. Lift and wave drag of hydrofoils.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 351–364. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.
The hydrofoil is replaced by a substitution vortex placed at the quarter-chord point and its image system in

the free surface. Lift and drag coefficients are calculated and compared with certain experimental results.
L. M. Milne-Thomson (Madison, Wis.).

★ **Breslin, John P. A simplified theory for the thrust deduction force on a body of revolution.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 337–350. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

Approximate calculations are made for the drag augmentation, on a body of revolution, arising from a lightly loaded propellor of finitely-many blades. The results are essentially the same as those for propellors of infinitely-many blades for very small loadings. The dependence of the effect on slope and curvature of the sectional-area curve at the stern, axial clearance and propeller-diameter-to-body-length ratio is investigated.
R. C. MacCamy (Pittsburgh, Pa.).

Polášek, Jan. Annular blade in axially-symmetric flow. Apl. Mat. **1** (1956), 334–375. (Czech. Russian and English summaries)

The author has previously found an approximate solution to the problem of an annular blade in axi-symmetric flow. It was published in a paper with this same title [Výzkumná Zpráva VÚTS **VT-Z 5326**]. In this paper the author presents a much more accurate theory. He solves the equations controlling the phenomenon and obtains: a) an expression for the distribution of vortex rings on the blade in terms of a trigonometric series; b) formulas for two components of the primary velocity multiplied by the distance from the axis of symmetry in terms of trigonometric series. The recurrence relation for the coefficients of the last series is very complicated. A numerical example shows that the computational work is considerable, even if one retains just a few terms of the series. It seems that a numerical solution on a high-speed digital computer is preferable. *T. Leser* (Aberdeen, Md.).

Schmidt, F. H. On the diffusion of heated jets. Tellus **9** (1957), 378–383.

General expressions are obtained for a two dimensional heated jet that is forced to leave an infinitely long aperture; analogous expressions are also given for the axially symmetric jet. A new velocity profile is proposed which, for both the two dimensional and axially symmetric cases, gives good agreement with experiment.
M. H. Rogers (Shrivenham).

Yamada, Hikoji. On the approximate expression of Stokes waves. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **5** (1957), 143–155.

These waves are waves of finite amplitude on deep water. At the free surface the pressure is constant. By suitable transformations of hodograph type, the field of flow is transformed into the inside of a circle. On the circumference the complex velocity satisfies a non-linear boundary condition. Two approximate methods of solution are considered. The first is an iterative scheme for the boundary values, applied at a finite number of points; the second replaces the boundary condition by T. V. Davies's non-linear boundary condition [Proc. Roy. Soc. London. Ser. A. **208** (1951), 475–486; MR **13**, 396], for which an explicit solution can be found. The two methods are applied to the 'highest wave' (having an angle $2\pi/3$ at the crest), and to a wave of small steepness. For the latter the two methods agree very well, for the former the agreement is less satisfactory. *F. Ursell.*

★ **MacCamy, Richard C. Motion of ships of shallow draft.** University of California, Institute of Engineering Research, Berkeley, Calif., 1956. ii+28 pp.

In investigating the motion of a ship in surface water it is profitable to assume the motions small (linearization) and that they are two dimensional (in a vertical plane cutting the ship in half lengthwise). Often (not here) the ship is also assumed to be very narrow, in which case one obtains, to lowest order (ship=knife edge), an explicit solution [A. S. Peters and J. J. Stoker, Comm. Pure Appl. Math. **10** (1957), 399–490; MR **19**, 703]. In the present paper the author considers instead the case of a ship of small draft, i.e., a ship approximating a thin raft. Also, the ship is not to be underway in the waves. By expanding formally with respect to the draft one obtains a sequence of "dock" type problems [K. O. Friedrichs and H. Lewy, Comm. Appl. Math. **1** (1948), 135–148; MR **10**, 336], which, however, are not solved explicitly. A study of the behavior of the solution near the edges of the ship shows that the two terms of the pressure remain finite there, but that the velocity becomes infinite. Also, the first two terms of the formal solution are shown asymptotic to the exact solution. Finally, numerical solutions to "dock" type problems are discussed. *W. Littman.*

Gouyon, René. Sur le transport de masse dans l'onde de Gerstner. C. R. Acad. Sci. Paris **245** (1957), 2181–2184.

For Gerstner waves it is shown that there is a net mass flow, in the direction opposite to that of propagation, across any vertical element which remains submerged throughout the motion. It is verified that this is balanced by an opposite flow across vertical elements which are only intermittently submerged. *G. B. Whitham.*

★ **de Neufville, Albert. The dying vortex.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 365–375. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

In the case of a straight vortex, axially symmetric plane flow, the Navier-Stokes equation is solved for the rotation z in terms of Laguerre's polynomials of the independent variable $x=r^2/4\nu t$ in the form

$$z=\frac{{t_0}^n}{t^{n+1}}\,e^{-x}L_n{}^0(x).$$

The set of solutions represents vortices consisting of a core and n layers of alternating sense of rotation. The circulation is zero at infinity for all solutions except when $t\to\infty$. Since the solutions constitute a complete orthogonal set, it is possible to represent, at some instant t_0, a Helmholtz vortex as a convergent infinite series of the above solutions, if it is possible to create a Helmholtz vortex at all. Such a vortex will therefore decay. The results of this paper appear to the reviewer to be new and important. *L. M. Milne-Thomson* (Madison, Wis.).

Mallick, D. D. Secondary flow in a rotating elliptic pipe. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 178–190.

A long straight pipe of elliptic section is made to rotate with constant angular velocity Ω about the minor axis of one of the cross-sections. If viscous liquid is flowing along the pipe under pressure, there is a secondary motion of the liquid in any cross-section. Then, the relation between the total rate of flow and the pressure gradient depends on Ω. A solution correct to order Ω^2 illustrates the general character of the secondary motion

and determines approximately the rate of flow due to a given pressure gradient. *W. R. Dean* (London).

★ **Guilinger, Willis H.; and Saibel, Edward. Numerical methods of solution of the adiabatic slider bearing without side leakage.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 285–297. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

Naumova, L. G. Three-dimensional laminar boundary layer on an infinite circular sector. Prikl. Mat. Meh. 21 (1957), 478–485. (Russian)

The geometric situation is as follows. Let there be a line vortex along the vertical z-axis, and let the horizontal x, y plane between $\theta=0$ and $\theta=\theta_0$ (in polar coordinates) represent a sector of a plate cut from a circle of infinite radius. What is the flow pattern resulting from the presence of this plate? The fluid is assumed viscous but incompressible. (Evidently, if $\theta=0$ is the "leading edge" where the flow enters, a boundary layer is built up which increases in thickness as θ increases.)

The differential equations are linearized in the usual way by expanding the dependent variables in powers of a small parameter. The "zeroth" approximation corresponds to the undisturbed vortex flow if no plate is present. The "first" approximation is then a perturbation of the "zeroth" approximation, resulting from the insertion of the plate (i.e., consideration of terms in the first power of the parameter). The differential equations for the first approximation are solved using Laplace transforms. *R. B. Davis* (Syracuse, N.Y.).

Hasimoto, Hidenori. Boundary-layer slip solutions for a flat plate. J. Aero. Sci. 25 (1958), 68–69.

A series expansion for the solution of the von Mises-transformed equation with wall condition $u=h\partial u/\partial y$. The author gives the first two terms, confirming that the shear stress τ_0 remains finite at $x=0$, while the no-slip (Blasius) solution yields $\tau_0=O(x^{-\frac{1}{2}})$. *G. Kuerti.*

Morgan, G. W.; Pipkin, A. C.; and Warner, W. H. On heat transfer in laminar boundary-layer flows of liquids having a very small Prandtl number. J. Aero. Sci. 25 (1958), 173–180.

For very small Prandtl number (as in liquid metals), the viscous boundary layer is much narrower than the thermal one, so that there is essentially potential flow over most of the thermal layer. The first two terms of a series representation of the temperature for arbitrary potential flow and wall-temperature distribution are obtained, and several typical problems are explicitly solved in this way. (From the authors' summary). *G. Kuerti.*

Sparrow, E. M. Application of Görtler's series method to the boundary-layer energy equation. J. Aero. Sci. 25 (1958), 71–72.

The b.l. energy equation for an incompressible heat-conducting fluid in plane motion (a) and the b.l. equation of motion for the cross-flow component in the flow about a yawing general cylinder (b) have the same structure except for an additional term in (b). The author notes that (a) and (b) become identical in Görtler's independent variables [J. Math. Mech. 6 (1957), 1–66; MR 18, 843], which permits the solution of a class of heat-transfer problems of type (a) in terms of Görtler's series.

G. Kuerti (Cleveland, Ohio).

Wood, W. W. The asymptotic expansions at large Reynolds numbers for steady motion between non-coaxial rotating cylinders. J. Fluid Mech. 3 (1957), 159–175.

This is a problem which provides a guide to the determination of higher order boundary layer corrections, particularly in closed motions. The stream function is obtained as a perturbation expansion $\sum_{n=0}^{\infty}\psi_n\gamma^n$ in the parameter, γ, which depends on the (small) displacement of the cylinder axes. The expansion appears to be uniformly valid with respect to the Reynolds number, and the asymptotic form for large Reynolds number of ψ_1 in particular is found. This agrees with the independent determination of ψ_1 from boundary layer theory.

H. C. Levey (Melbourne).

★ **Guha, C. R.; and Yih, Chia-Shun. Laminar convection of heat from two-dimensional bodies with variable wall temperatures.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 29–40. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

Using the velocity distribution given by a Blasius-Howarth expansion for the boundary layer on an arbitrarily shaped two-dimensional body, a series of functions are integrated numerically to obtain the steady-state temperature in the boundary layer for a variable wall temperature. *R. C. DiPrima* (Troy, N.Y.).

Reid, W. H. On the stability of viscous flow in a curved channel. Proc. Roy. Soc. London. Ser. A. 244 (1958), 186–198.

The author re-examines the problem of the stability of viscous flow in a curved channel by means of some new methods developed by Chandrasekhar and the present author. The criterion of instability obtained agrees with that given by Dean, but disagrees with that given more recently by Yih and Sangster. The flow pattern is found to be of the familiar cellular type, but with a marked asymmetry. *C. C. Lin* (Cambridge, Mass.).

★ **Görtler, Henry. On forced oscillations in rotating fluids.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 1–10. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

For small forced oscillations in an inviscid incompressible fluid rotating with uniform angular velocity ω about an axis, the author pointed out years ago that the pattern of flow would be of the wave type if the angular frequency β of the oscillation is lower than 2ω. In this paper the author reports some recent progress in this study, including a description of some theoretical results and their experimental confirmation. An analogy is drawn with the forced oscillations in a fluid with stable density stratification. *C. C. Lin* (Cambridge, Mass.).

Kraichnan, Robert H. Relation of fourth-order to second-order moments in stationary isotropic turbulence. Phys. Rev. (2) 107 (1957), 1485–1490.

The validity of the assumption that the second and fourth-order moments of the velocity amplitude distribution in homogeneous turbulence are related as for a normal distribution (quasi-normality) is examined by deriving consequences for the equilibrium range of Fourier components of the motion. The energy equation for a

single realization is

$$(1)\qquad (\partial/\partial t+2\nu k^2)[\tfrac{1}{2}u_i(\underset{\sim}{k},t)u_i^*(\underset{\sim}{k},t)]=\sum_{\underset{\sim}{k}'} Q(\underset{\sim}{k},\underset{\sim}{k}'),$$

where $Q(\underset{\sim}{k},\underset{\sim}{k}')=\mathrm{Imag}\{-k_l u_i^*(\underset{\sim}{k}',t)u_l^*(\underset{\sim}{k}-\underset{\sim}{k}',t)u_i(\underset{\sim}{k},t)\}$ is the rate of transfer of energy from the Fourier component of wave-number k' to that of wave-number k, and $u_i(\underset{\sim}{k},t)$ is a Fourier component of the motion. Necessarily, (2) $Q(\underset{\sim}{k},\underset{\sim}{k}')+Q(\underset{\sim}{k}',\underset{\sim}{k})=0$ .By integrating (1), it is shown that: (a) condition (2) is not satisfied if quasi-normality is assumed for the two-time four-point velocity distribution; and (b) energy is not conserved within the inertial sub-range if quasi-normality is assumed for the two-time two-point distribution. The reasons for the failure of these apparently reasonable assumptions are discussed, and an explanation is found in terms of a weak statistical dependence of Fourier components of different wave-numbers which permits the total contribution of the skew moments (neglected by hypotheses of quasi-normality) to be comparable with the contribution of the non-skew moments by virtue of their immensely greater number. It is suggested that the more restricted assumption of quasi-normality of the simultaneous amplitude distribution may also prove to be inconsistent with the equations of motion. In an appendix, the same type of argument is used to demonstrate inconsistencies in the theory of turbulence recently formulated by Chandrasekhar [Proc. Roy. Soc. London. Ser. A. **229** (1955), 1–19; MR **16**, 968].

A. A. Townsend (Cambridge, England).

Beckers, H. L. Heat transfer in turbulent tube flow. Appl. Sci. Res. A. **6** (1956), 147–190.

The problem considered is the transfer of heat from the walls of a smooth circular tube by a fully developed turbulent flow. Initially the fluid has a temperature T_i, but from the section $z=0$ the wall temperature changes to T_0 and the fluid in the tube approaches that temperature as z increases. If the convective effects of the turbulent motion may be represented by an eddy conductivity, the partial differential equation for the mean temperature may be solved by separation of variables, giving the temperature distribution as a linear combination of eigenfunctions. Using distributions of stream velocity and eddy conductivity obtained from approximate formulae for the velocity distributions in the "laminar" sub-layer, in the transition layer and in the fully turbulent core, the first two eigenvalues and eigenfunctions are calculated, and temperature distributions and heat transfers from the walls are derived for several conditions and compared with the available experimental material.

A. A. Townsend.

Corrsin, Stanley. Statistical behavior of a reacting mixture in isotropic turbulence. Phys. Fluids **1** (1958), 42–47.

The effect of isotropic turbulence on a passive scalar field is generalized to include certain types of chemical reactions. It is assumed that the chemical reaction has no effect on the velocity field, and that the reaction rate is proportional to an integral power of the local concentration. First and second order reactions are considered, using mean values at one and two points. The resulting equations exhibit the usual indeterminacy, and further approximations are then introduced to obtain some simple estimates of the decay rates for the mean square concentration fluctuations. *W. H. Reid.*

Kusukawa, Ken-ichi. On the two-dimensional compressible flow over a thin symmetric obstacle with sharp shoulders placed in an unbounded fluid and in a choked wind tunnel. J. Phys. Soc. Japan **12** (1957), 1031–1041.

The flow is assumed to separate at the shoulder as a sonic streamline and then to maintain the free stream direction (zero slope) to infinity after it is attained (Roshko wake model). The boundary-value problem is solved by application of Imai's method (erroneously called WKB) of correspondence between compressible and incompressible flows. The method is not valid for mixed flow fields. Numerical results for a wedge nose agree fairly well at Mach number one with calculations based on Tricomi equation with other downstream conditions, as well as with the solution of the same problem for the Tricomi equation by Helliwell and Mackie [J. Fluid Mech. **3** (1957), 93–109; MR **19**, 914]. The same flow between parallel walls is called choked. The relation between choked and free-flight flows agrees very well with that computed by Marschner on the basis of the Tricomi equation. *J. D. Cole* (Pasadena, Calif.).

Kusukawa, Ken-ichi. On the compressible flow over a slender body of revolution with a flat base placed in an unbounded fluid and in a choked wind tunnel. J. Phys. Soc. Japan **12** (1957), 1042–1048.

A method of correspondence between axially symmetric and two-dimensional transonic flow previously derived by the author [same J. **12** (1957), 401–419; MR **19**, 90] is applied to obtain approximate flows about a finite paraboloid, a finite cone, and body with radius $\sim$ (distance from nose)^2. The two-dimensional flow is computed approximately by the method of the paper reviewed above. Reviewer notes that the correspondence method is based on arbitrarily omitting a term in the equation of motion; the addition of a further approximate method makes an assessment of the results difficult. Results for the body with (radius) $\sim$ (distance)^2 are obviously of novalue. *J. D. Cole* (Pasadena, Calif.).

Yang, Hsun-Tiao. Reduction of Ikenberry-Truesdell equations to Burnett equations for slip flow. J. Aero. Sci. **25** (1958), 404–405.

The author verifies in detail the result derived in § 15 of E. Ikenberry and C. Truesdell, J. Rational Mech. Anal. **5** (1956), 1–54 [MR **17**, 796]. The result which the author attributes to Schaaf and Sherman is given in § 17 of the paper cited. *C. Truesdell* (Bloomington, Ind.).

Gustafson, W. A.; and v. Krzywoblocki, M. Z. On multiplicity theorems and an exact solution in diabatic flow. II. Acta Phys. Austriaca **11** (1957), 294–320.

Kiebel has given an exact solution of plane isentropic flow with rotation, which contains both subsonic and supersonic regions [Prikl. Mat. Meh. **11** (1947), 193–198; MR **9**, 541; **11**, 752]. That result is here generalized to a one-parameter family of flows with heat addition. The analysis is first carried out using the flow equations in conventional form, paralleling Kiebel. A much more elaborate analysis of the process of separation of variables is required. The same solutions are then deduced from the canonical form of the flow equations given in part I [Acta Phys. Austriaca **11** (1957), 131–146; MR **19**, 604]. That procedure is simplified by first showing that for the special form of heat addition considered, the gas follows a polytropic law, so that the problem is reduced to one

solved by Prim [J. Rational Mech. Anal. **1** (1952), 425–497; MR **14**, 107]. *M. D. Van Dyke.*

★**Liepmann, H. W.; and Roshko, A. Elements of gasdynamics.** Galcit aeronautical series. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1957. xv+439 pp. $11.00.

Of the current books on gas dynamics, this one is most likely to survive the transition to the space age. Liepmann and Puckett's well-known "Introduction to aerodynamics of a compressible fluid" [Wiley, New York, 1947 [MR **9**, 111] has been completely rewritten. The result is slightly more advanced (and inevitably somewhat thicker), but remains a lucid account of the fundamentals of gas dynamics. Some special topics now mostly of historical interest have been dropped in favor of unified treatments, for example of the similarity rules of compressible flow. Such new topics in high-speed flow as molecular vibration and dissociation are included, starting from the admirable first chapter on "concepts from thermodynamics". A final chapter on "concepts from gas kinetics" surveys the essentials of the kinetic theory, touching on transport properties, rarefied gas flows and relaxation effects, and concluding with an exhliarating and individualistic view of the limits of continuum theory. *M. D. Van Dyke.*

Kočina, N. N. Some exact solutions of equations of one-dimensional unsteady motion of a perfect gas. Prikl. Mat. Meh. **21** (1957), 449–458. (Russian)

★**Gispert, H.-G. Numerische Behandlung eines nichtlinearen Variationsproblems aus der Gasdynamik.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 113–118. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

The irrotational, stationary subsonic flow around circular and elliptic cylinders is investigated using a variational method which starts out similarly to the approach used by Braun [Ann. Physik (5) **15** (1932), 645–676] but which retains the nonlinear character of the problem. The unknown velocity potential ϕ is assumed to have the form $\phi=\phi_1+\phi_2$, where ϕ_1 is the potential for the incompressible flow and $\phi_2(r^{-1},\theta)=f_1(r^{-1})\cos\theta+f_3(r^{-1})\cos 3\theta+f_5(r^{-1})\cos 5\theta$ is the additional potential in polar coordinates. The unknown functions f_i are determined by 3 nonl inear differential equations of the second order which are solved numerically. Numerical results are given for the circular cylinder and Mach numbers $M=0.35$ and 0.4 which almost concide with the values given by I. Imai [Proc. Phys.-Math. Soc. Japan **23** (1941), 180–193] but required less computation. The case of an elliptic cylinder for $M=0.6$ and ratio 0.2 of the principal axis is briefly discussed. *U. W. Hochstrasser* (Lawrence, Kan.).

Belotserkovsky, O. M. Flow past a circular cylinder with a detached shock wave. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 509–512. (Russian)

Let r, θ be polar coordinates. Let $r=1+\varepsilon(\theta)$ be the equation of the shock in plane supersonic non-viscous flow about the circle $r=1$. On each ray $\theta=$constant let r_i, θ divide the segment between circle and shock into N equal parts. Approximate the various flow functions by polynomials of the form $u(r,\theta)=\sum_0^N a_m(\theta)(r-1)^m\varepsilon^{-m}(\theta($ which assume the values u_i at r_i, θ. Substitute in the equations of plane flow, integrate with respect to r, and use the boundary and shock conditions to obtain a system of $3N$ first order ordinary differential equations for the velocity components $u_i(r_i,\theta)$, v_i, streamfunction ψ_i, and ε, and $N-1$ finite equations for $\phi_i(\psi_i)=p_i/\rho_i$. This system has been solved on the BESM computer to determine the subsonic and part of the supersonic region of flow. Those initial values not known a priori must be adjusted to remove singularities at the sonic line. No attempt has been made to prove convergence as $N\to\infty$. However, pressure distributions on the circle calculated for $N=2$ and $N=3$ at Mach number 3.0 are almost indistinguishable from experimental data. *J. H. Giese.*

Chushkin, P. I. Sonic gas flow past ellipses. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 517–519. (Russian)

Summary of Prikl. Mat. Meh. **21** (1957), 353–360 [MR **19**, 1006]. *J. H. Giese* (Havre de Grace, Md.).

★**Murthy, S. N. B. On the method of characteristics for supersonic flows with shock waves.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 131–140. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

The author is mistaken in suggesting that the form of the method valid for homentropic flow is applicable to heterentropic flow behind curved shocks. For proper, and intelligible, treatments, cf. the standard texts. *R. E. Meyer* (Providence, R.I.).

★**Handbook of supersonic aerodynamics. Vol. 3. Section 6.** Navord Report 1488. U.S. Government Printing Office, Washington, D. C., 1957. iv+94 pp. (not consecutively paged) $1.50.

Brief accounts of the assumptions and results of the principal methods of calculating pressure distributions on two-dimensional aerofoils are given, with a very little reference to the effects of the boundary layer. An extremely large number of graphs indicate the results of applying the method to particular cases, as well as giving a few comparisons with experiment. *M. J. Lighthill.*

Kogan, Abraham. An application of Crocco's stream function to the study of rotational supersonic flow past airfoils. Quart. J. Mech. Appl. Math. **11** (1958), 1–23.

The theory of small plane perturbations from a uniform supersonic stream is developed on the basis of Crocco's stream function. In contrast to previous treatments based on a velocity potential, this permits consideration of flows involving rotation, as from a strong curved shock wave. The details are set forth for the second approximation, and applied to two situations. First, consideration of a thin airfoil disturbing a uniform flow leads to Busemann's second-order solution for surface pressures, but gives also the second approximation for the whole flow field including the bow shock wave. Second, the basic flow is taken to be that behind the oblique shock on a wedge having the same leading-edge angle as an airfoil of interest. The first perturbation reproduces Crocco's results [Aerotecnica **17** (1937), 519–534] for the intial gradients, and the second approximation gives the initial curvatures of flow quantities at the leading edge. Comparison with exact numerical solutions for a biconvex airfoil at Mach numbers of 10 and ∞ shows that the resulting three

terms of a Taylor series closely represent the surface pressures over the first third of the chord.

M. D. Van Dyke (Los Altos, Calif.).

Iordanskiĭ, S. V. On the stability of a plane stationary shock wave. Prikl. Mat. Meh. **21** (1957), 465–472. (Russian)

Consider a one-dimensional flow with a plane shock wave produced by impulsively started uniform motion of a plane piston into quiescent gas. Slightly perturb it to obtain a non-steady two dimensional flow with pressure $p+\pi(x, t)e^{iky}$, shock equation $x=-v_0t+\zeta(x, t)e^{iky}$, etc. To first order, π satisfies the wave equation (*) $\partial^2\pi/\partial t^2-c^2\partial^2\pi/\partial x^2+k^2c^2\pi=0$. Boundary conditions can be found at the piston and shock, while initial values may be prescribed arbitrarily. The solution π can be expressed in terms of the Riemann function of (*) and boundary values of π and its derivatives. At boundary points, this yields an integro-differential-difference equation for the boundary values of $d\zeta/dt=\partial\zeta/\partial t-v_0\partial\zeta/\partial x$, knowledge of which enables one to complete the construction of $\pi(x, t)$. The author considers first a piston infinitely far from the shock. By means of Laplace transform theory, he shows that for various choices of parameters of the undisturbed flow, $d\zeta/dt$ either grows exponentially as $t\to\infty$, or else $d\zeta/dt=0(t^{-0.5n})$, where $n=0$, 1, or 3. Some of these conclusions disagree with earlier results of D'yakov [Ž. Eksper. Teoret. Fiz. **27** (1954), 288–295; MR **16**, 537], whose perturbations were, however, also harmonic in t. The author makes similar estimates for flows with pistons at finite distances from the shock. *J. H. Giese.*

Černyĭ, G. G. Adiabatic motions of a perfect gas with shock waves of great intensity. One-dimensional non-steady motions. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 3, 66–81. (Russian)

Consider the Lagrangian form of the equations for unsteady symmetrical plane, cylindrically symmetrical, or spherically symmetrical unsteady isentropic flow. The author expands the pressure $p(m, t)$, density $\rho(m, t)$, and space coordinate $R(m, t)$ in series $R=R_0+\varepsilon R_1+\cdots$, $p=p_0+\varepsilon p_1+\cdots$, $\varepsilon\rho=\rho_0+\varepsilon\rho_1+\cdots$, respectively, where $\varepsilon=(\gamma-1)/(\gamma+1)$. He assumes that a shock moves on the curve $R=R_0(t)$, and determines the five functions $R_1, \cdots, \rho_1$ in terms of quadratures involving R_0 and its derivatives of first and second order. The constants of integration are chosen to satisfy the shock conditions. Applications to two problems are considered: (i) the flow produced by a "piston" $R=R^*(t)$; (ii) point explosions. Each leads to an integro-differential equation for $R_0(t)$, for which approximate solutions are found. The author carries out calculations by his methods, and obtains good agreement with known exact solutions or with results calculated by other methods. *J. H. Giese* (Havre de Grace, Md.).

Kubansky, P. N. Axisymmetrical acoustical streaming near a resonator. Akust. Ž. **3** (1957), 337–341. (Russian)

Using Westervelt's [J. Acoust. Soc. Amer. **25** (1953), 60–67; MR **14**, 700] theory for acoustic streaming, an analytical verification is made of earlier experimental observations [Ž. Tehn. Fiz. **22** (1952), 1483–1493] on streaming near the mouth of a flanged cylindrical cavity resonator excited by high amplitude sound waves. Computed flow lines for a 13.6 kc resonator show a mushroom-like flow pattern centered on the cavity.

W. W. Soroka (Berkeley, Calif.).

Khaskind, M. D. Diffraction and radiation of acoustic waves in liquids and gases. I. Akust. Ž. **3** (1957), 348–359. (Russian)

The author's earlier [Ž. Eksper. Teoret. Fiz. **16** (1946), 634–646; MR **8**, 545] theory of radiation damping and virtual mass experienced by a harmonically oscillating rigid or deformable body in an infinite compressible fluid is extended to include a superimposed sound field of the same frequency. Use is made of asymptotic characteristic functions of radiation and scattering. Plane, spherical and cylindrical sound fields are considered. Calculated curves are shown for oscillating sphere, cylinder and circular plate. A brief development for random plane acoustic waves is given. *W. W. Soroka.*

Oestreicher, Hans L. Field of a spatially extended moving sound source. J. Acoust. Soc. Amer. **29** (1957), 1223–1232.

The problem considered is that of a sound source at rest in a uniform stream. The fluid is assumed to be a perfect gas and the flow around the source to be potential flow. The author starts his discussion with a general second order homogeneous equation which follows from these assumptions. This equation can then be "solved" in terms of the point source function. When the radiating source is a rigid surface, the result is remarkably simple in appearance. The resulting expression can then be expanded in a multipole expansion in which the point source function and its higher derivatives appear. *H. Feshbach.*

Oestreicher, Hans L. Representation of the field of an acoustic source as a series of multiple fields. J. Acoust. Soc. Amer. **29** (1957), 1219–1222.

The field of an arbitrary acoustic source is given in terms of an infinite series of multipole fields in which the coefficients are given in terms of surface integrals of the amplitude and its normal derivative evaluated on the surface. The convergence of the series is discussed.

H. Feshbach (Cambridge, Mass.).

Wells, C. P.; and Leitner, A. A Lebedev transform and the "baffle" problem. Quart. Appl. Math. **15** (1958), 430–434.

The problem of a circular disc of radius a, vibrating in an infinite rigid baffle, is solved by means of the Lebedev transform. The expression for the velocity potential in the form of an integral transform of the above type is transformed into series expansions valid for $r\leqq a$ and $r\geqq a$, respectively, in terms of Legendre and spherical Bessel functions. *F. Oberhettinger* (Madison, Wis.).

★**Srivastava, A. C. Superposability in non-Newtonian fluids.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 187–194. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

The author determines necessary and sufficient conditions for superposability of two flows of non-Newtonian fluids. For constitutive equations, he uses quadratic relations of the Reiner-Rivlin types. In particular, he finds that any two irrotational motions are superposable.

J. L. Ericksen (Baltimore, Md.).

★**Pai, S. I. One-dimensional unsteady flow of magneto-gasdynamics.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 251–261. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

L'auteur étudie les caractéristiques du système d'équations aux dérivées partielles qui régit les écoulements rectilignes d'un fluide compressible doué de viscosité, de conductivité thermique et de conductivité électrique. Lorsque la viscosité et la conductivité thermique sont nulles, tandis que la conductivité électrique est soit nulle soit infinie, le système est hyperbolique; dans tous les autres cas le système est parabolique. *H. Cabannes.*

Trehan, S. K. The stability of an infinitely long cylinder with a prevalent force-free magnetic field. Astrophys. J. **127** (1958), 436–445.

L'auteur étudie la répartition du champ magnétique à l'intérieur d'un cylindre de révolution rempli d'un fluide incompressible doué d'une conductivité électrique infinie; sur la paroi du cylindre, le champ magnétique est supposé parallèle aux génératrices. Le cylindre étant remplacé par une surface de révolution à méridienne sinusoidale, l'auteur étudie l'influence de cette modification sur l'énergie spécifique totale; il y a instabilité lorsque la longueur d'onde de la méridienne dépasse une certaine valeur critique. *H. Cabannes* (Marseille).

Neuringer, Joseph L.; and McIlroy, W. Incompressible two-dimensional stagnation-point flow of an electrically conducting viscous fluid in the presence of a magnetic field. J. Aero. Sci. **25** (1958), 194–198.

Une plaque infinie étant placée dans un fluide incompressible doué de conductivité électrique, les auteurs étudient l'action d'un champ magnétique extérieur normal à la plaque; cette action se traduit par une réduction importante de la traînée. Ce résultat doit pouvoir être utilisé dans le cas d'un obstacle se déplaçant dans l'atmosphère à une vitesse hypersonique; il existe en effet entre l'onde de choc détachée et l'obstacle une région dans laquelle l'air est ionisé et par suite doué de conductivité électrique. *H. Cabannes* (Marseille).

Carini, Giovanni. Osservazioni sulle onde cilindriche della magneto-idrodinamica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 482–488.

The propagation of cylindrical waves in an infinite homogeneous medium with a finite electrical conductivity (σ) and with a prevalent uniform magnetic field in the z-direction is considered. The perturbation in the field is in the transverse, θ, direction and is proportional to

$$J_1(kr)e^{-\nu z}\cos(\alpha z-\omega t),$$

where J_1 denotes the Bessel function and k, ν, α and ω are constants. These latter constants are related by

$$k^2=\nu^2-\alpha^2+2V^2\alpha\nu/a^2\omega,$$

$$V^2(\nu^2-\alpha^2)-2a^2\alpha\nu\omega+\omega^2=0,$$

where $V=(B_0^2/\rho\mu)^{\frac{1}{2}}$ denotes the Alfvén velocity; $a=c/(\mu\sigma)^{\frac{1}{2}}$; c is the velocity of light and μ is the magnetic permeability. *S. Chandrasekhar* (Williams Bay, Wis.).

Kemp, Nelson H. On hypersonic stagnation-point flow with a magnetic field. J. Aero. Sci. **25** (1958), 405–407.

A spherical nose is considered with a radial magnetic field. Following Lighthill's treatment of the non-magnetic problem [J. Fluid Mech. **2** (1957), 1–32; MR **19**, 352], the flow between the detached shock wave and body is assumed incompressible (but rotational). For small magnetic Reynolds number (low electrical conductivity) an approximate solution is found for the vicinity of the stagnation point. Predicted reductions in heat transfer are slight under practical conditions. *M. D. Van Dyke.*

Kidder, R. E. Unsteady flow of gas through a semi-infinite porous medium. J. Appl. Mech. **24** (1957), 329–332.

The author considers the non-steady, one-dimensional flow of a gas through a semi-infinite porous medium; the problem is to study a non-linear partial differential equation in two independent variables. This equation is further reduced to a non-linear, ordinary differential equation in one independent variable by proper choice of the independent variable. A power-series solution of the resulting equation is discussed. The author computes coefficients of the zeroth-order, first-order, and second-order terms in the expansion, for various values of the independent variable. A comparison is made of the zeroth-order term and the solution of the corresponding linear equation. *N. Coburn* (Ann Arbor, Mich.).

Mikhailov, G. K. Percolation in a rectangular cofferdam when the capillary rise is very high. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 725–728. (Russian)

The filtration process of the present paper is considered to obey Darey's well-known law. In order to obtain solutions which satisfy the necessary boundary-conditions, the author employs conformal mapping theory. By virtue of these transformations, the author is able to solve the problem completely in terms of elliptic functions. An exact expression is also obtained for the filtrational discharge. This discharge is represented in a figure, together with curves based on the solutions of other authors. *K. Bhagwandin* (Oslo).

See also: Numerical Methods: Dumitrescu, Ionescu and Toth. **Probability:** Scheidegger. **Statistics:** Craddock; Charnock; Rushton and Neumann. **Mechanics of Particles and Systems:** Ogorodnikov. **Elasticity, Plasticity:** Eirich; Yamamoto. **Optics, Electromagnetic Theory, Circuits:** Trehan. **Relativity:** Taub. **Astronomy:** Auluck and Kothari; Javorskaya; Lidov. **Geophysics:** Lauwerier; Saltzman.

Optics, Electromagnetic Theory, Circuits

★**Conrady, A. E. Applied optics and optical design.** Dover Publications, Inc., New York, 1957. v+518 pp. $2.95.

It is probably correct to say that most, if not all, designers of symmetrical optical systems will need to have this well known book on their shelves. Though in some ways rather old-fashioned, it is still the only one of its kind. The volume is divided into ten chapters: I. Fundamental Equations; II. Spherical Aberration; III. Physical Aspect of Optical Images; IV. Chromatic Aberration; V. Design of Achromatic Object Glasses; VI. Extra- axial Image Points; VII. The Optical Sine Theorem; VIII. Trigonometrical Tracing of Oblique Pencils; IX. General Theory of Perfect Optical Systems; X. Ordinary Eyepieces. The author has been extraordinarily careful to be explicit and elementary, and to limit the mathematical equipment required of the reader to an absolute minimum.

This is no doubt one of the reasons for the popularity of this work. Except for Chapter III, the treatment is based entirely on purely geometrical optics, and the theory rests directly upon the laws of refraction. (The idea of a characteristic function does not seem to occur anywhere.) Broadly, the underlying procedures consist of the judicious combination of the results of paraxial and primary aberration theory on the one hand, and those afforded by trigonometrical ray tracing on the other. The topics introduced are all dealt with in the most painstaking detail, and a large number of explicit numerical examples is provided throughout.

On the negative side, this work is hardly suitable as an introduction to optical theory, even in its limited context. The author's algebraic manipulations are, on the whole, rather clumsy and lead him — to take just one example — to make the fantastic statement (p. 108) that the coefficient of tertiary spherical aberration 'would utterly defy analytical determination' [cf. Buchdahl, J. Opt. Soc. Amer. **46** (1956), 941–943; MR **19**, 355]. The appearance of a reprint of this work at the present time may, however, serve as a useful reminder of the dichotomy which exists between practical and academic optics. The industrial designer of camera lenses, say, largely shuns the latter, and, many claims to the contrary, he may indeed best be served by the work under review. *H. Buchdahl* (Hobart).

Bergstein, Leonard. General theory of optically compensated varifocal systems. J. Opt. Soc. Amer. **48** (1958), 154–171.

The author gives the general theory of a system of lenses containing n parts, which move against each other in such a way that the focal length of the over-all system changes between two desired values, and that during the movement the image plane of the over-all system coincide n times with a desired reference plane, deviating from it during the movement by a designed small amount.

The data of the lens system is obtained by a set of equalities and inequalities, where the latter can be used to minimize the spacings between the movable parts and to minimize the maximum displacement of the focal plane.

M. Herzberger (Rochester, N.Y.).

Back, Frank G.; and Lowen, Herbert. Generalized theory of Zoomar systems. J. Opt. Soc. Amer. **48** (1958), 149–153.

A Zoomar is a special type of varifocal optical system. A few general design principles for Zoomar systems and a brief account of their history are given.

E. W. Marchand (Rochester, N.Y.).

Vandakurov, Yu. V. The theory of aberrations of electron-optical systems with a curvilinear axis. Ž. Tehn. Fiz. **27** (1957), 1850–1862. (Russian)

This paper develops the theory of aberrations of electron-optical systems based on the method of trajectories, which the author considers to be far superior to the eikonal method. Basic equations are derived for the calculations of geometric aberrations of the second and third orders and for chromatic aberrations of the first order, The equations, derived for the relativistic case, are valid for arbitrary electric and magnetic fields. Explicit formulas are obtained for the aberration and also for dispersion for different velocities and masses. Conditions are found for the absence of geometric aberrations of the second order. *J. E. Rosenthal* (Passaic, N.J.).

Bernard, Michel-Yves; et Hue, Jean. Aberration d'ouverture des lentilles à forte convergence. C. R. Acad. Sci. Paris **244** (1957), 732–735.

This note is a continuation of same C. R. **243** (1956), 1852–1854 [MR **19**, 91] on the aberration of strong focusing lenses. In the present note, the authors have examined the deviations of the trajectories from the Gaussian paths for electrostatic and also for magnetic lenses when third order aberrations are taken into account. The equations representing the deviation of the image point from the Gaussian image are computed for an isogenic incident bundle and the third order deviations are expressed in terms of the aberration coefficients of the lens (electrostatic or magnetic type). The aberration coefficients are then computed for a special magnetic lens and the corresponding characteristic image curves (spots) in the focal planes are plotted. *N. Chako* (Flushing, N.Y.).

Akhiezer, A. I.; Prokhoda, I. G.; and Sitenko, A. G. Scattering of electromagnetic waves in a plasma. Soviet Physics JETP **6** (1958), 576–582.

Plasma oscillations of frequency $\Omega=\sqrt{(4\pi n_0 e^2/m)}$ are associated with oscillations in the dielectric "constant" of the medium. Consequently, the propagation in the plasma of an electromagnetic wave of frequency $\omega_0>\Omega$ is accompanied by propagation at frequencies $\omega=\omega_0\pm n\Omega$, where n is an integer. For a plane wave of frequency ω_0, the angular distribution of the energy of the scattered wave is computed for $n=\pm 1$ in the presence and absence of a uniform magnetic field. *A. A. Blank.*

Totaro, Carmelo. Una osservazione sulle condizioni al contorno dell'elettrodinamica dei corpi in moto. Boll. Un. Mat. Ital. (3) **12** (1957), 609–611.

The author treats boundary conditions for the electromagnetic field of moving bodies. He starts from Minkowski's equations for the electromagnetic field and applies these to boundaries between 2 different media. Upon some regrouping and calculations the boundary conditions are shown to be: $B_n^{(1)}=B_n^{(2)}$ and $D_n^{(2)}-D_n^{(1)}=\omega$. Obviously the latter equation differs from the well-known case of non-moving bodies. *M. J. O. Strutt.*

Lampariello, Giovanni. Intorno alla propagazione di onde elettromagnetiche nei superconduttori e ai teoremi di reciprocità del campo elettromagnetico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 302–310.

The author sets out to study wave propagation in superconducting media and to obtain a reciprocity theorem valid for them. He starts from the theory of superconductive media as set forth by F. London and by M. von Laue. In essence, the ordinary electromagnetic field equations are supplemented by an additional current density vector called supercurrent density. Assuming all space vectors to depend on time t by the factor $\exp(i\omega t)$, the equations of the electromagnetic field are derived on the above basis, leading up to two four-dimensional vectors Ω and Γ, which satisfy the eqs. $\Box\Omega=-\mu_0\Gamma$ and $\operatorname{div}\Gamma=0$. It is then shown, that an equivalent of Green's well known three-dimensional theorem is valid for these four-dimensional vectors. *M. J. O. Strutt* (Zürich).

Wentzel, Donat G. Motion of charged particles in a force-free magnetic field. Astrophys. J. **126** (1957), 559–564.

The motion of charged particles in a force-free magnetic

field (curl $\mathbf{H}=\alpha\mathbf{H}$) with axial symmetry and α=constant is considered. From the angular momentum integral (which follows from the axial symmetry of the field), the regions of inaccessibility for particles of given energy are defined. Further, by evaluating the drifts of the guiding center, it is shown that particles cannot escape across surfaces where the radial component of the magnetic field vanishes. Finally, it is shown that when allowance is made for radiation losses by the synchrotron mechanism, $v_\perp^2/\mathbf{H}$ is a constant, where $v_\perp$ is the velocity transverse to the field. *S. Chandrasekhar* (Williams Bay, Wis.).

Trehan, S. K. On the stability of force-free magnetic fields. Astrophys. J. **126** (1957), 429–456; errata 601.

Force-free magnetic fields are those for which the magnetic field $\mathbf{H}$ is everywhere parallel to curl $\mathbf{H}$. A particular force-free configuration is one which, within a prescribed radius R, satisfies the condition curl $\mathbf{H}=\alpha\mathbf{H}$ where α is a determinate constant; and which outside the radius R is a vacuum field (i.e. curl $\mathbf{H}=0$) and tends to a uniform field (of intensity B in a certain direction $\mathbf{1}_z$) at infinity. This field is given by

$$\mathbf{H}_i=\mathbf{1}_r\frac{\partial}{\sin\theta\partial\theta}\left(\sin^2\theta P\right)-\mathbf{1}_\theta\frac{\sin\theta}{r}\frac{\partial}{\partial r}(r^2P)+\mathbf{1}_\phi\alpha r\sin\theta\mathbf{P} \qquad (|\mathbf{r}|\leqq R)$$

$$\mathbf{H}_0=\mathbf{1}_rB\left(1-\frac{R^3}{r^3}\right)\cos\theta-\mathbf{1}_\theta B\left(1+\frac{R^3}{2r^3}\right)\sin\theta \quad (|\mathbf{r}|\geqq R)$$

where $\mathbf{1}_r$, $\mathbf{1}_\theta$ and $\mathbf{1}_\phi$ are unit vectors along the three principal directions in a spherical system of coordinates r, θ and ϕ; also

$$\mathbf{P}=\frac{3B}{2}\frac{R^{\frac{1}{2}}}{\alpha}\frac{J_{3/2}(\alpha r)}{J_{3/2}'(\alpha R)} \text{ and } J_{3/2}(\alpha R)=0,$$

where $J_{3/2}(x)$ is the Bessel function of order 3/2. Let such a magnetic field pervade a gaseous medium of uniform density ρ and ratio of specific heats γ. The stability of such a medium for axisymmetric perturbation is considered. A variational procedure is derived for solving the characteristic frequencies of the system. By determining when these frequences become complex, the author shows that the system is unstable if

$$R^2\geqq\frac{3}{2\pi\rho G_1}\left[1\cdot1\frac{B}{4\pi\rho}+c^2\right],$$

where c $(=(\gamma p(\rho)^{\frac{1}{2}})$ denotes the velocity of sound. The foregoing criterion is the equivalent of Jeans's condition for gravitational instability of an infinite homogeneous medium. *S. Chandrasekhar* (Williams Bay, Wis.).

Lenz, Hanfried. Geradlinige Potentialfelder. Jber. Deutsch. Math. Verein. **60** (1958), Abt. 2, 39.

Es wird gezeigt, das die Potentialfelder des idealen Platten-, Zylinder- oder Kugelkondensators die einzigen mit geraden Feldlinien sind.

Aharoni, A.; and Shtrikman, S. Magnetization curve of the infinite cylinder. Phys. Rev. (2) **109** (1958), 1522–1528.

This paper presents an extensive calculation of the magnetization of an infinite ferromagnetic cylinder. The reversal of magnetization in such a cylinder does not occur by "rotation in unison" of the spins, but by more complicated processes. In particular, when the axis of the cylinder is parallel to the field, at a certain negative field intensity the saturated state along the axis ceases to be stable. This field is called "nucleation field", and is the smallest eigenvalue of a certain set of partial differential equations. The corresponding eigenfunctions are the modes of deviation from the initially saturated state. This eigenspectrum is rigorously calculated in the present paper and it is shown that for cylinders with "large" radius the mode is magnetization curling, while for those with small radii it is something closely related to magnetization buckling. The transition from curling to buckling is abrupt. It is also shown that all other modes give higher nucleation fields. The magnetization reversal occurs in one jump, which means that the magnetization curve for the cylinder is a rectangular loop. *M. J. Moravcsik.*

Giovanelli, R. G. Radiative transfer in discontinuous media. Austral. J. Phys. **10** (1957), 227–239.

This paper investigates the diffusion of radiation through a model medium consisting of randomly distributed spherical regions of uniform absorption and scattering coefficients κ and σ separated by free space. The equation of radiative transfer (in integral form) is obtained by first considering how an arbitrary radiation field incident on a particular spherical region is scattered by it and then expressing the radiation field in the source free region as the superposition of such scattered radiations. Approximate solutions are obtained when the following conditions are fulfilled: 1) the spherical regions are of sufficient optical thicknesses to ignore the light directly transmitted through them and 2) the fraction of the volume occupied by the regions may be considered small. Geometrical shadows and "reflection" by relatively small spherical regions are the only effects which are considered in detail though the theory is formulated in more general terms.

Approximate expressions are obtained for the reflectance of a semi-infinite medium as a function of κ/σ and of the size and the mean separation of the regions.

S. Chandrasekhar (Williams Bay, Wis.).

Grosjean, C. C. On a new approximate one-velocity theory of multiple scattering in infinite homogeneous media. Nuovo Cimento (10) **4** (1956), 582–594.

In an earlier paper [Nuovo Cimento (10) **3** (1956), 1262–1275; MR **18**, 627] the author has shown that in an infinite isotropically scattering medium, the density ρ_s of diffusely scattered neutrons satisfies, in a certain approximation, the differential equation

$$(1)\qquad \nabla^2\rho_s-\frac{3}{\lambda^2}\frac{1-\omega}{2-\omega}\rho_s(\mathbf{r})+\frac{3\omega}{(2-\omega)v\lambda}S(\mathbf{r})=0,$$

where ω is the albedo for single scattering; λ is the mean free path; v is the velocity of the particle; and $S(\mathbf{r})$ represents an isotropic distribution of sources. Combining this approximate differential equation with the exact equation

$$(2)\qquad \operatorname{div}[\mathbf{j}-(1-\omega)\mathbf{j}_d]=\omega S(\mathbf{r})-(1-\omega)v\rho_s(\mathbf{r})/\lambda,$$

which the total current $\mathbf{j}$ and that $\mathbf{j}_d$ due to the particles which have been scattered more than once satisfy, one can obtain the integral

$$(3)\qquad \mathbf{j}=(1-\omega)\mathbf{j}_d-\tfrac{1}{3}(2-\omega)\lambda v\operatorname{grad}\rho_s.$$

For a medium scattering radiation with a slight degree of anisotropy, the result (3) is generalized to give the formula

$$(4)\qquad \mathbf{j}=(1-\omega)\mathbf{j}_d-\frac{[2-\omega+(1-\omega)^2\langle\cos\gamma\rangle]\lambda v}{3(1-\omega\langle\cos\gamma\rangle)}\operatorname{grad}\rho_s,$$

where γ is the angle of scattering and $\langle\cos\gamma\rangle$ is the average

value taken with respect to the phase function.

The paper is concerned largely with comparisons of equations (3) and (4) with the results of exact solutions in cases where these are known. *S. Chandrasekhar.*

Theimer, O.; and Plint, C. A. Light scattering by nearly perfect crystals. Ann. Physics 3 (1958), 408–422.

The paper deals with the theory of light scattering by transparent ionic crystals containing imperfections, such as vacancies and impurities. The concentration and configuration of these is discussed. Thermal scattering, scattering by macroscopically homogeneous distribution of point imperfections and scattering by bad regions in the neighbourhood of dislocation lines are treated mathematically. The application of the theory shows that NaCl crystals have a dislocation density of 10^7–10^9 cm^{-2}, an average concentration of vacancies of 10^{15}–10^{17} cm^{-3}, and bad regions with an average diameter of about 1000 Å and an excess vacancy concentration of about 10^{18} cm^{-3}.

V. Vand (University Park, Pa.).

Barford, N. C.; and Reynolds, G. T. Graphical determination of the path of a scattered particle. Nuovo Cimento (10) 4 (1956), 929–932.

Given a charged particle entering a flat plate at a point A, travelling at an angle φ_1 with respect to the normal to the plate and leaving the plate at point B and angle φ_2; the authors then compute the probability that half way through the plate the particle is at a distance l_0 away from AB and that the projected velocity of the particle makes an angle θ_0 with respect to line AB. From this the most probable l_0 and θ_0 can be determined. By repeated application of this result it becomes possible to approximate the most probable path more and more closely.

H. Feshbach (Cambridge, Mass.).

Rubinowicz, A. Ein bisher nicht beachteter Fall, in dem der Kirchhoffsche Ansatz zur angenäherten Beschreibung der Beugungserscheinungen versagt. Acta Phys. Polon. 17 (1958), 13–20. (Russian summary)

This is an essay on some of the limitations on the Kirchoff approximations in diffraction theory.

A. E. Heins (Pittsburgh, Pa.).

Leonard, D. J.; and Yen, J. L. Junction of smooth flared wave guides. J. Appl. Phys. 28 (1957), 1441–1448.

The present treatment of junctions of smooth flared waveguides is an extension of A. F. Stevenson's [same J. 22 (1951), 1447–1460; MR 13, 802] investigation of wave guides of arbitrary cross section. The latter is not valid for junctions if the derivatives are not continuous along the axis of the wave guide. The authors start from a field representation in a smooth wave guide by means of suitable infinite series expressions. They then proceed to the consideration of a junction of two smooth wave guides and set forth the conditions prevailing at the junction. From these, a matching procedure is obtained, which allows one to calculate the series coefficients at the junction. From these expressions, the reflection coefficients of the junction of straight rectangular and circular wave guides to various flared rectangular and circular wave guides are evaluated and the results are illustrated by curves. Some remarks are made as to the comparison of the theory with experimental values. *M. J. O. Strutt* (Zürich).

Belevitch, Vitold. Sur quelques problèmes d'approximation de Tchebycheff en théorie des télécommunications. Bull. Soc. Math. Belg. 8 (1956), 158–168.

To solve Tchebycheff approximation problems involving rational functions, the author is led to a parametric representation by Jacobi's elliptic functions. Simple analytic function theory arguments show the representation to be a natural extension of the parametric representation of the Tchebycheff polynomial $y=Tn(x)$ by $y=\cos nu$ and $x=\cos u$ and justify the particular form of the elliptic functions involved.

Some applications to filters and polyphased systems are indicated. *G. G. Weill.*

Mayeda, Wataru; and Seshu, Sundaram. Topological formulas for network functions. Univ. of Illinois Engrg. Exper. Station Bull. no. 446 (1957), 35 pp.

This is an essentially expository representation confined to networks without mutuals. The classical results on, in general, non-planar connected networks: mesh and node transformations involving incidence and circuit matrices, trees and cord sets, are reviewed. Topological formulas for one- and two-terminal networks, the latter in terms of the element admittances (node analysis), are derived in terms of multi-tree and multi-cord-set product discriminants; copious use is made of Percival's complementarity relation. The results are used to obtain topological expressions for all the significant parameters of T- and π-networks. The dual representation of two-terminal networks in terms of element impedances follows. Applications to network synthesis are treated briefly, and present gaps are pointed out. Illustrative examples are given throughout. *H. G. Baerwald* (Cleveland, Ohio).

Feuerstein, E. Intermodulation products for ν-law biased wave rectifier for multiply frequency input. Quart. Appl. Math. 15 (1957), 183–192.

The integral

$$A=\int_C (iu)^{-(\nu+1)}e^{-iuB}\prod_{r=0}^{n} J_{m_r}(P_r u)\,du,$$

where C is the entire real u-axis with a downward indentation of radius δ at the origin, gives, apart from a constant factor, the intermodulation coefficients resulting from passage of the signal $V(t)=\sum_{r=0}^{N} P_r\cos(p_r t+\gamma_r)$ through a ν-law wave rectifier of bias B. A study of the Laurent series in δ, obtained by elementary operations on the integral, enables the author to tabulate for all cases of ν forms of A which are rapidly convergent and amenable to numerical analysis. *R. N. Goss.*

Bolinder, E. Folke. General method of analyzing bilateral, two-port networks from three arbitrary impedance measurements. Ericsson Technics 14 (1958), 3–37.

The impedance transformations arising in general bilateral, two-port networks are described in terms of two fixed points and a multiplier. Stereographic mappings and analytical methods are used parallel, illustrated by numerical examples. *G. Kron* (Schenectady, N.Y.).

★ **Ollendorff, Franz. Technische Elektrodynamik. Bd. II. Innere Elektronik. Teil 2. Elektronik freier Raumladungen.** Springer-Verlag, Wien, 1957. xii+620 pp. $23.35

Es handelt sich beim vorliegenden Buch um eine theoretische Behandlung der Raumladungsfelder im weitesten Sinne, jedoch mit Ausschluss der Ionenfelder. Im einleitenden Kapitel bringt Verf. die phänomenolo-

gische Grundlagen: Die Gleichungen der Geschwindigkeits-Verteilungen, die Elektronenemission und die Hochvakuumdiode. Das zweite Kapitel befasst sich im wesentlichen mit Raumladungen in Dioden und Elektronenstrahlröhren. Das dritte Kapitel behandelt die Elektronensteuerung in der Triode und wird von einem vierten Kapitel über die Elektronensteuerung in Mehrgitterröhren gefolgt. Das nächste Kapitel behandelt die Raumladungsschwingungen (Monotron, Bremsfeldröhre, Erregung von Hohlraumresonatoren). Im folgenden Kapitel wird die magnetische Steuerung der Raumladungsfelder behandelt, welche in den verschiedenen Magnetron-Typen auftritt. Das letzte Kapitel behandelt endlich die Wendelröhren und die Mischstrahl-Verstärker, sowie die Plasma-Wellen. Die Behandlung der Probleme in diesem Buch stützt sich weitgehend auf eigene Arbeiten des Verf. und es ist darum nicht überall leicht zu sehen, ob und wie die Ergebnisse sich in die aus der Literatur bekannten Arbeiten eingliedern lassen. *M. J. O. Strutt* (Zürich).

See also: Linear Algebra: Cederbaum. **Partial Differential Equations:** Miranker; Keller. **Elasticity, Plasticity:** Saitô and Kato. **Fluid Mechanics, Acoustics:** Trehan; Carini. **Quantum Mechanics:** Nozières and Pines. **Relativity:** Pham Mau Quan; Lampariello. **Astronomy:** Auluck and Kothari; Javorskaya.

Classical Thermodynamics, Heat Transfer

Cowan, Robert D. Properties of the Hugoniot function. J. Fluid Mech. 3 (1958), 531–545.

Using the first and second laws of thermodynamics, the author gives several properties of the Hugoniot function $H(p, v)=E(p, v)-E(p_0, v_0)+\frac{1}{2}(p+p_0)(v-v_0)$ and the Hugoniot curves $dH=(Tds+dA)=0$ which are derived through essentially geometrical arguments. Apparently, the author's motivation for this paper was the desire to develop a feeling for the geometrical relations between the initial assumptions and the final deductions, many of which have been obtained previously analytically. Two problems in which the application of these geometrical properties have been found of value are included.

C. G. Maple (Ames, Iowa).

Golubenkov, V. N. Heat convection in a rotating circular tube with constant temperature gradient. Prikl. Mat. Meh. 21 (1957), 439–440. (Russian)

A solution is obtained for the velocity and temperature distributions for laminar flow. The results involve the radial and axial positions, with parameters involving the angular velocity which produces a centrifugal force, and the sectional angular momentum of the fluid in the tube.

N. A. Hall.

Camia, Frédéric. Courbes balistiques de la chaleur (et de la diffusion) dans un cylindre de rayon fini et dans le cas d'un flux radial, équations fondamentales. C. R. Acad. Sci. Paris 245 (1957), 2218–2221.

The classical solution for the temperature function $\theta(r, t)$ in a hollow cylinder of infinite length, initially at temperature $\theta=f(r)$, with simple homogeneous conditions on the cylindrical boundaries, is derived formally. Certain limiting cases are noted.

R. V. Churchill.

Chow, Tse-Sun. On a problem of heat conduction with time-dependent boundary conditions. Z. Angew. Math. Phys. 8 (1957), 478–484.

The boundary value problem $u_t=a^2\nabla^2 u$ $(0\leq\rho<r, 0\leq\phi\leq 2\pi, |z|<s, t>0)$, $u+h_1u=f_1(\phi, z)\alpha_1(t)$ on $\rho=r$, $u_z+h_2u=f_2(\rho, \phi)\alpha_2(t)$ on $z=s$, $u_z-h_3u=f_3(\rho, \phi)\alpha_3(t)$ on $z=-s$, $u=u_0(\rho, \phi, z)$ when $t=0$, is solved in series form for temperatures $u(\rho, \phi, z, t)$ in a cylinder. The author applies a method of resolving the problem into problems in which time t does not enter explicitly in the boundary conditions. Orthogonal functions are used in solving the component problems. *R. V. Churchill.*

Jung, H. Zur Theorie der gesteuerten Anheizvorgänge. Z. Angew. Math. Mech. 38 (1958), 56–69. (English, French and Russian summaries)

A strictly formal perturbation procedure is applied to the nonlinear partial differential equation $\lambda_0(1+\lambda_1 T)\nabla^2 T=c\gamma\partial T/\partial t$, where λ_0, λ_1, c and γ are constants. The boundary condition also contains a coefficient that is a linear function of the temperature T. The author gives no explanation of the connection between his partial differential equation and the heat equation for a medium whose conductivity depends on temperature, nor of his assumption that the initial temperature is a harmonic function. He applies the results of his manipulative methods to some examples in order to indicate the influence of the first perturbation term in the formula for T.

R. V. Churchill (Ann Arbor, Mich.).

Singh, S. N. Heat transfer by laminar flow in a cylindrical tube. Appl. Sci. Res. A. 7 (1958), 325–340.

The problem of heat transfer, for fully developed Poiseuille flow in a circular tube, has been worked out exactly, including the effects of axial heat conduction, viscous dissipation and constant heat flow through the wall. The wall of the tube is kept at a constant temperature and the fluid enters at a different temperature. The temperature variations within the fluid are assumed to be small. The eigenfunctions of the problem are expressed in terms of an infinite series whose typical term is a product of an exponential function and Bessel function of order zero. The first four eigenvalues are computed by an iterative method. It is concluded that the effect of axial heat-conduction is almost negligible for high Peclet number ($Pe>100$). Also, the heat transfer problem of potential flow is given and same conclusion on axial heat conduction is found.

It is mentioned that there is a similar work by K. Millsaps and K. Pohlhausen [Proc. Confer. on Differential Equations, Univ. of Maryland, 1956, pp. 271–294; MR 18, 538]. *L. N. Tao* (Chicago, Ill.).

Siegel, R.; Sparrow, E. M.; and Hallman, T. M. Steady laminar heat transfer in a circular tube with prescribed wall heat flux. Appl. Sci. Res. A. 7 (1958), 386–392.

The heat transfer problem in a circular tube with a fully developed laminar velocity profile at the entrance and a prescribed heat flux at the wall is studied, based on the assumptions that the longitudinal heat conduction and viscous dissipation may be ignored. The first part of the paper gives a solution of a uniform wall heat flux problem using the fully developed temperature distribution as the boundary condition at infinity. The solution is given in an infinite series. Its first seven eigenvalues and eigenfunctions are evaluated on a digital computer. Then by the method of superposition the solution

of non-uniform wall heat flux problem is shown by an integral containing the series solution of the uniform wall heat flux problem. There is no numerical calculation other than those mentioned. *L. N. Tao* (Chicago, Ill.).

See also: Partial Differential Equations: Douglas. **Fluid Mechanics, Acoustics:** Beckers.

Quantum Mechanics

Salecker, H.; and Wigner, E. P. Quantum limitations of the measurement of space-time distances. Phys. Rev. (2) **109** (1958), 571–577.

The authors examine the limitations imposed on the measurement of space-time intervals by quantum mechanics, since this is a necessary preliminary to the study of the problem of whether the gravitational field of atomic systems and elementary particles is observable in principle. As measuring rods are macrophysical objects, only clocks and light-signals (single quanta) are used. It is assumed that the ultimate macrophysical signal recorder is so far distant that it need not be regarded as part of the microphysical clock. Because of the requirement that the light signal does not spread out in all directions, the authors restrict their analysis to a world of one spatial dimension only. (To the reviewer this seems to be the main criticism of this important pioneer investigation.) The principal result established is that if the clock is to have an accuracy τ and a running time $T=n\tau$, then its mass must exceed $n\hbar/c^2\tau$, in order that the time at which the photon strikes it be predetermined to within τ. The uncertainty (spread) in this mass is given by $\hbar/c^2\tau$.

In the later sections of the paper, examples of microphysical clocks are suggested. The authors favour a composite clock comprising k 'pointers', broadly analogous to the pointers of ordinary watches. Each 'pointer' consists of a single particle oscillating to and fro between two other particles. The periods are $n_1\tau$, $n_2\tau$, $\cdots$, and the accuracies are τ, $n_1\tau$, $\cdots$, respectively, the kth pointer having period T and the accuracy of the period of the previous pointer. The pointers must be locked so that the motion of the second is controlled by that of the first, and so on. It is found that the total mass and the mass uncertainty of this clock are reasonably close to the respective lower limits deduced from general principles, provided that the possibility is admitted of constructing states whose wave functions are Gaussian wave-packets.
G. J. Whitrow (London).

Nozières, Philippe; and Pines, David. Electron interaction in solids. General formulation. Phys. Rev. (2) **109** (1958), 741–761.

D. Pines und D. Bohm [Phys. Rev. (2) **85** (1952), 338–353; **92** (1953), 609–625] haben auf Grund ihrer theoretischen Untersuchungen die Wechselwirkungen in einen Plasma (Dichtes Elektronengas, das durch darin enthaltene positive Ionen ladungskompensiert ist.) in zwei Teile zerlegt. Die Wechselwirkungen grosser Reichweite verursachen eine kollektive Bewegung des Plasmas, dessem Frequenz um die von L. Tonks und I. Langmuir [ibid. **33** (1929), 195–210] hergeleitete Frequenz eines Elektronengases $\omega_p=(4\pi e^2N/m)^{\frac{1}{2}}$ herum liegt. Übrig bleiben noch die Wechselwirkungen kurzer Reichweite, welche die individuellen Bewegungen der Elektronen verursachen.

Ziel der vorliegenden Arbeit ist diese Methode auf feste Körper anzuwenden. Die erste Komplikation entsteht bei denen dadurch, dass in einem festen Körper ein periodisches Potentialfeld vorhanden ist und deshalb auch Übergänge innerhalb eines Energiebandes und auch zwischen solchen Banden auftreten können; die zu den letzteren gehörenden Frequenzen werden mit ω_{n0} bezeichnet. Dann können zwei Fälle auftreten, entweder ist $\omega_{n0}\gg\omega_p$. Das ist ein recht seltener Fall, kann jedoch in festen Edelgasen bestehen. Der zweite $\omega_{n0}\ll\omega_p$ ist der weitaus häufigste. Das Kriterium, dass in einem festen Körper Plasmaschwingungen mit der Frequenz ω_p ($\hbar\omega_p$ ist die Plasmonenergie bzw. ein Plasmon der Verfasser) auftreten, ist eine grosse elektronische Polarisierbarkeit.

Den Ausgangspunkt der Berechnungen bildet die Hamiltonsche Funktion der Valencelektronen

$$(1)\qquad H=\sum_k\left\{\frac{p_i^2}{2m}+V(\mathfrak{r}_i)\right\}+\sum_k{}'\frac{2\pi e^2}{k^2}\,(\rho_k\rho_{-k}-N),$$

wo N die Zahl der Elektronen, $V(r_i)$ die von dem Gitten der Atomkerne und der ungestörten Elektronen herrührende potentielle Energie und das letzte Glied die von den Dichtenschwankungen der Elektronen herrührende Energie bedeutet. $\rho_k=\sum_i\exp(-i\mathbf{k}\cdot\mathfrak{r}_i)$. In folgenden wird die Wechselwirkung von Plasmonen und Elektronen besprochen, danach werden neue kollektive Koordinaten eingeführt und mit Hilfe einer kanonischen Transformation wird die erwähnte Wechselwirkung beseitigt. Danach wird die Konvergenz des Verfahrens untersucht und zuletzt wird noch die Dämpfung der Plasmonen und die Wechselwirkung von denen mit den abgeschlossenen Ionenschalen besprochen. Ein sehr wichtiges Ergebnis der Berechnungen ist, dass da die Plasmonenschwingungen normalerweise nicht angeregt sind, (Ihre Energie beträgt meistens ca. 15 eV.) so genügt es bei den meisten festen Körpern das vereinfachte Problem H_0+H_{sr} zu berücksichtigen, wobei

$$H_0=\sum_i\left\{\frac{p_i^2}{2m}+V(\mathfrak{r}_i)\right\}$$

[Vgl. (1)] und H_{sr} der bloss von den Kräften kurzer Reichweite herrührende Teil der Hamiltonschen Funktion ist. *Th. Neugebauer* (Budapest).

Nozières, Philippe; and Pines, David. Electron interaction in solids. Collective approach to the dielectric constant. Phys. Rev. (2) **109** (1958), 762–777.

In der oben referierten Arbeit der Verfasser wurde die Theorie von D. Pines und D. Bohm bezüglich der Wechselwirkungen in einem Plasma auf das Problem der festen Materie übertragen. Ziel der vorliegenden Untersuchung ist auf Grund dieser theoretischen Resultate eine strengere Theorie der Dielektrizitätskonstante auszuarbeiten. Die Theorie dieser Konstante ist nämlich trotz ihrer scheinbaren Einfachkeit ein verwickeltes und schon viel besprochenes Problem. Dabei müssen die Fragen beantwortet werden: Wann hat eigentlich der Begriff der Dielektrizitätskonstante einen Sinn, welche Bedeutung hat das Lorentzsche innere Feld und wie lautet die quantenmechanische Formel für die Dielektrizitätskonstante?

Zur Einleitung wird die bekannte theoretische Herleitung der Dielektrizitätskonstante von H. A. Lorentz besprochen. Mit α bezeichnen die Verfasser die mikroskopische Polarisierbarkeit, welche durch die Gleichung $\mathbf{P}=\alpha\mathbf{E}_M$ definiert wird. ($\mathbf{P}$ ist die Polarisation und $\mathbf{E}_M$ bedeutet das lokale Feld im Inneren des festen Körpers; meistens bezeichnet man diese Grösse mit $N\alpha$, wo dann N

die Zahl der Atome in der Volumeneinheit und α die Polarisierbarkeit eines Atoms ist.) Die Lorentz-Lorenzsche Formel hat dann die Form $4\pi\alpha=3(\varepsilon-1)/(\varepsilon+2)$. Die Sellmeyersche Formel lautet dagegen $4\pi\alpha=\varepsilon-1$. Das Gültigkeitsbereich dieser Formeln ist ein sehr altes Problem.

Im leeren Raum schreiben die Verfasser die Wechselwirkung von zwei Ladungen in der Form $\sum_k \frac{1}{2}M_k{}^2\gamma_k\gamma_{-k}$; in einem festen Dielektrikum muss diese Formel noch mit ε dividiert werden. Den Ausgangspunkt der weiteren Betrachtungen bildet die Hamiltonsche Funktion

$$(1)\quad H_{ges}=H_0+H_{st}+\sum_{k<k_c}\{\tfrac{1}{2}|M_k\rho_k+P_k|^2+\tfrac{1}{2}M_k{}^2\gamma_k\gamma_{-k}+(M_k-\mu_k)\gamma_{-k}(M_k\rho_k+P_k)+\mu_kM_k\gamma_{-k}\rho_k\}.$$

H_0 und H_{st} sind in dem zitierten Referat definiert, ρ bezieht sich auf die Ladungen der übrigen Elektronen, P auf das Feld der Plasmonen und μ_k ist eine noch willkürliche Konstante. (1) wird zuerst mit Hilfe einer kanonischen Transformation umgeformt und dann werden durch eine weitere solche Transformation die Wechselwirkungsglieder beseitigt. Ein sehr wichtiges Ergebnis der Verfasser ist, dass auch wenn $4\pi\alpha\ll 1$ ist und auch wenn $4\pi\alpha\gg 1$ ist (Metalle) die quantenmechanische Formel

$$\varepsilon(k)=1+4\pi\alpha(k)=1+\frac{4\pi e^2}{m}\sum_n\frac{f_{n0}(k)}{\omega_{n0}{}^2}$$

gültig ist. Problematisch ist nur das Gebiet $4\pi\alpha\lesssim 1$. (Lorentzscher Isolator.)

Im folgenden wird noch die Frequenzabhängigkeit der Dielektrizitätskonstante die Frequenzabhängige longitudinale Leitfähigkeit und die optischen Eigenschaften der festen Körper hergeleitet. *Th. Neugebauer* (Budapest).

Pursey, D. L. Invariance properties of Fermi interactions. Nuovo Cimento (10) **6** (1957), 266–277.

The zero (observable) mass of the neutrino leads to the dynamical independence of the projections $\psi_1=\frac{1}{2}(1+\sqrt{5})\psi$ and $\psi_2=\frac{1}{2}(1-\sqrt{5})\psi$ of the neutrino field. This, together with the fact that the only interactions in which the muon and electron fields occur linearly are the weak Fermi interactions, leads to the invariance of the physical predictions under arbitrary independence gauge transformations of the electron and muon fields as well as the two projections of the neutrino field. These transformations on the fields are equivalent to transformations on the coupling constants. The equivalence of various sets of coupling constants c_i, c_i' of the Fermi interaction related by these transformations is shown to imply that only the scalar $(c_jc_k{}^*+c_j'c_k'{}^*)$ and pseudoscalar $(c_jc_k'{}^*+c_j'c_k{}^*)$ bilinear "invariants" enter into the predictions of the theory (evaluated in lowest order). The conditions for T, C, P invariances are restated in terms of these bilinear combinations, and certain additional results (mainly computational simplifications) are pointed out.

E. C. G. Sudarshan (Rochester, N.Y.).

Prange, R. E.; and Pratt, R. H. Applications of a high-energy Coulomb wave function. Phys. Rev. (2) **108** (1957), 139–143.

An asymptotic expansion for the high energy Coulomb wavefunction for an electron which differs from the free Dirac spinor by a space-dependent phase factor only is obtained by expansion of this phase in inverse powers of the electron momentum and retaining only the lowest term. This function ψ (obtained from a WKB approximation by earlier authors) is compared with the more exact function ψ_{SM} of Bethe and Maximon [Phys. Rev. (2) **93** (1954), 768–784; MR **15**, 919]. It is shown that ψ leads to a correct expression for the total cross section for atomic photoelectric effect in the high energy limit; the computation is far simpler than one using ψ_{SM}.

E. C. G. Sudarshan (Rochester, N.Y.).

van Kampen, N. G. Causality and the Kramers-Kronig relations. I. Nederl. Tijdschr. Natuurk. **24** (1958), 1–14. (Dutch)

This paper is a non-technical introduction to the theory of dispersion relations (Kramers-Kronig relations in the terminology of the author). The discussion is entirely classical and can perhaps best be summarized with the aid of the author's own words: "Het gevolg kan niet aan de oorzaak voorafgaan". (In rough translation: The response can never precede the driving force.) If a force $X(t)$ causes a response $x(t)$ in a linear system according to the formula $x(t)=\int_{-\infty}^{+\infty}G(t-t')X(t')dt'$, the causality principle says $G(t)=0$ for $t<0$. As is well-known, this implies certain analyticity properties for the Fourier transform of the function $G(t)$. The author illustrates this principle with the aid of various examples from the theory of elasticity, a damped oscillator with a driving force, electric networks, the propagation of light through a dielectric medium etc. The figure "I" in the title of the paper makes it a reasonable assumption that one can expect a second part with, perhaps, quantum mechanical applications. *G. Källén* (Lund).

McLennan, J. A., Jr. Conformal invariance and conservation laws for relativistic wave equations for zero rest mass. Nuovo Cimento (10) **3** (1956), 1360–1379.

The author's summary reads: It is shown that the relativistic wave equations obtained by Gårding [Proc. Cambridge Philos. Soc. **41** (1945), 49–56; MR **6**, 283] for particles of zero rest mass and arbitrary spin are invariant under the conformal transformations of space-time. Conservation laws corresponding to this invariance are obtained. They are formally identical to those obtained by Bessel-Hagen for the electromagnetic field [Math. Ann. **84** (1921), 258–276] with the exception of those found for the scalar field. For this case, the conservation laws contain extra terms which arise from the fact that the trace of the energy-momentum tensor is not zero.

A. J. Coleman (Toronto, Ont.).

McLennan, James A., Jr. Improper Lorentz transformations. Phys. Rev. (2) **109** (1958), 986–989.

The theory of representations of the improper, homogeneous Lorentz group allows of irreducible representatives of the space and time reflection operators satisfying

$$P^2=1,\quad T^2=\pm 1,\quad PT=\pm TP.$$

The representations commonly used in physics do not exhaust these essentially distinct alternatives. The unusual representations would describe, for instance, a particle with two parity states. At present it seems unlikely that any of the known "elementary" particles, except possibly the neutrino, should correspond to one of these unusual representations. *J. C. Taylor.*

Gerjuoy, E. Outgoing boundary condition in rearrangement collisions. Phys. Rev. (2) **109** (1958), 1806–1814.

The boundary conditions on the solution to the non-relativistic Schrödinger equation $(H-E)\Psi=0$ for an arbitrary complex collision of a system of n particles is investigated. It is shown that the usual requirement for

the scattered wave ϕ to be everywhere outgoing means that ϕ behaves asymptotically like G, the outgoing Green's function for the total Hamiltonian. The above requirement also implies that a certain integral $I(G, \phi)$ vanishes if integrated over the surface at infinity in the $3n$-dimensional space of all particles. However $I=0$ does not necessarily imply that ϕ behaves asymptotically like G. It is concluded that for real energies E, the Lippmann-Schwinger integral equation need not have a unique solution. On the other hand, for complex energies $E=i\epsilon$ no boundary conditions are needed, and hence the solution of the integral equation is always unique. In general, results proved by operator techniques will apply to the solution of $(H-E)\Psi=0$ satisfying the boundary condition $I=0$, but will not apply to an arbitrary solution Ψ of the Lippmann-Schwinger integral equation for real energy. *M. J. Moravcsik* (Livermore, Calif.).

Kirzhnits, D. A. Quantum corrections to the Thomas-Fermi equation. Soviet Physics. JETP **5** (1957), 64–71.

This interesting paper investigates the corrections to the Thomas-Fermi and Thomas-Fermi-Dirac equations consequent on the allowance for non-commutativity of quantum operators. The Hartree-Fock approximation to the density matrix is written in operator notation and, for electrostatic interaction, transformed into an equation which generalizes the Thomas-Fermi or the Thomas-Fermi-Dirac equations, depending on whether exchange effects are not or are included. An ansatz for the density matrix suggested by the nondegenerate or degenerate Fermi gas is then used. The functions, whose arguments are operators, appearing in the transformed equation, are expanded in powers of $\hbar$, or equivalently, in terms of successively higher order commutators, and the resulting equation solved by successive approximations. Second and fourth order corrections to the energy of an atom are calculated, and it is shown that the Weizsacker approximation is too large by a factor of 9. It is stated that a calculation for argon gives a fourth order correction of 3%, so the author believes that the method should converge rapidly. *C. A. Hurst* (Adelaide).

Moses, H. E. The Kohn-Hulthén variational procedure for the scattering operator and the reactance operator. I. Elementary form of the variational principles. Nuovo Cimento (10) **5** (1957), supplemento, 120–143.

Variation principles involving the differential operators of the Schroedinger equation are obtained for the amplitude of the scattered wave and for the corresponding reactance operator. These principles are given explicitly for the radial equation with zero angular momentum, for one and three dimensional scattering problems and for electron scattering by an atom. Finally these principles are expressed in abstract form not involving any particular representation. *H. Feshbach* (Cambridge, Mass.).

Moses, H. E. The Kohn-Hulthén variational procedure for the scattering operator and the reactance operator. II. Procedures independent of the normalization of the trial functions. Nuovo Cimento (10) **5** (1957), supplemento, 144–158.

This is a continuation of the paper reviewed above in which the variational principles have been rendered homogeneous in the wave function by performing a minimization with respect to a scale parameter. The scale parameter is applied only to the addition to the unperturbed wave function. *H. Feshbach.*

Morita, M.; and Morita, R. Saito. Time-reversal invariance and beta-gamma angular correlation. Phys. Rev. (2) **107** (1957), 1316–1321.

The angular correlation functions between beta and gamma rays from oriented nuclei is evaluated for an allowed β-transition followed by a gamma ($2\rightarrow 0$) transition. If β interactions which do not preserve time-reversal invariance are included, a term of the type $[\mathbf{J}\cdot(\mathbf{p}\times\mathbf{k})][\mathbf{J}\cdot\mathbf{k}]^n$ ($n=1$ or 3) appears. Here $\mathbf{J}$=spin of nucleus, $\mathbf{p}$ and $\mathbf{k}$ are the momenta of the electron and gamma ray respectively. The constant of proportionality contains the product of the Gamow-Teller and Fermi matrix elements. Results are obtained for these cases in which the circular polarization of the gamma rays is and is not observed. *H. Feshbach* (Cambridge, Mass.).

Eyges, Leonard. Some nonseparable boundary value problems and the many-body problem. Ann. Physics **2** (1957), 101–128.

In this paper the author proposes a technique for solving a class of non-separable boundary value problems in which an amplitude ψ is zero on the bounding surfaces of a region in whose interior ψ satisfies the scalar Helmholtz equation, which can be higher dimensional. The boundary surfaces consist of several separate closed surfaces S_i. It is assumed that the solution of the boundary value problem when only one of these surfaces is present can be solved with the general solution $\psi_n^{(i)}$ for the ith surface. The method consists of expressing ψ as a linear combination

$$\psi=\sum_{i,n} A_n^{(i)}\psi_n^{(i)}.$$

The coefficients $A_n^{(i)}$ are determined by the requirement that the boundary conditions be met, not only on the ith surface, but also on the others. To carry this program out, it is necessary to be able to expand any $\psi_n^{(i)}$ in terms of any of the other $\psi_n^{(i)}$. The author discusses convergence of this procedure, which is obviously good when the interaction of the surfaces is small. A number of problems are discussed: (1) off center sphere inside a sphere; (2) scattering of a plane wave by two fixed spheres; (3) Schroedinger equation for two hard spheres in a finite box; (4) many-body problem for hard spheres, with only binary collisions important. Some comments are also made on the simple boundary perturbation theory.
H. Feshbach (Cambridge, Mass.).

Steinmann, O. Äquivalente periodische Potentiale. Helv. Phys. Acta **30** (1957), 515–520.

The author considers the one-dimensional Schroedinger equation with a periodic potential, and applies the theory of Hill's equation. The equivalent mass is defined by $1/m^*=d^2E/dk^2$, in which E is the constant potential part of the Schroedinger equation and k is the characteristic exponent of its solution. He proves that a continuum of equivalent periodic potentials exists for each given periodic potential. These equivalent potentials may be calculated by a series method, for which convergence is proved. *M. J. O. Strutt* (Zürich).

Young W. H.; and March, N. H. Perturbation theory in wave mechanics. Phys. Rev. (2) **109** (1958), 1854–1855.

Some comments are presented on the application of the Rayleigh-Schrödinger and Brillouin-Wigner perturbation methods and in particular it is pointed out that if H is the Hamiltonian of a system represented by a wave

function ψ_0 and a perturbation h is applied, an upper bound to the eigenvalue of the perturbed system is given by the expression

$$(\psi_0, [H+h]\psi_0)+\frac{(\psi_0, \bar{h}\psi_1)+(\psi_1, \bar{h}\psi_1)}{1+(\psi_1, \psi_1)},$$

where ψ_1 is the first order perturbed wave function in the Rayleigh-Schrödinger expansion and $\bar{h}=h-(\psi_0, h\psi_0)$.

A. Dalgarno (Belfast).

Barut, A. O. Distribution functions for noncommuting operators. Phys. Rev. (2) **108** (1957), 565–569.

The author considers the problem of finding joint distribution functions of noncommuting operators in quantum mechanics. Previous papers by Dirac [Proc. Cambridge Philos. Soc. **26** (1930), 376–385] and Wigner [Phys. Rev. (2) **40** (1932), 749–759] discuss a real phase space distribution function which suffers, however, from the fact that expectation values can be taken with only a restricted class of operators. The author instead defines a complex function in the following manner: let $\alpha_1, \cdots, \alpha_n$ be a set of n different complete commuting sets of operators with ortho-normal eigenfunctions $|\alpha_1'\rangle\cdots|\alpha_n'\rangle$ and $|\ \rangle$ an arbitrary state. Then the distribution function in the state $|\ \rangle$ is defined by $f(\alpha_1', \cdots, \alpha_n')=(|\alpha_1'\rangle(\alpha_1'|\alpha_2'\rangle, \cdots (\alpha_n'|)$. In general the order of the arguments in f matters. The expectation value of any operator of the form of products of powers, $G=\alpha_1{}^m\cdots$ is given by $\langle G\rangle=\sum_{\alpha'} G(\alpha_1', \cdots)f(\alpha_1', \alpha_2', \cdots)$, where the order of the arguments in f of this equation are to be the same as that in the operator G. Thus $\langle pq\rangle\neq\langle qp\rangle$ since $f(p, q)\neq f(q, p)$ (where q=coordinate and p=momentum). In fact it is shown that $f(q, p)=f^*(p, q)$, and hence the imaginary part of f, arises due to the commutation relations while the real part of f reduces, in the classical limit, to the classical probability density function. The dynamical equation of motion for f is found to be a Markoff-type stochastic equation. Expectation values of more complicated operators can presumably be found by first expanding into a power series.

R. Arnowitt (Syracuse, N.Y.).

Meligy, A. S. The wave functions in Coulomb fields. Nuclear Phys. **1** (1956), 610–618.

The radial equation for an electron of energy L and mass m moving in the Coulomb field of a nucleus of charge Ze is

$$(1)\qquad \frac{d^2W}{dr^2}+\left\{\frac{2m}{\hbar^2}\left(E+\frac{Ze^2}{r}\right)-\frac{l(l+1)}{r^2}\right\}W=0$$

($W=r$ times the radial factor of the wave function, and $l\hbar$ is the angular momentum of the state considered). By obvious transformations, (1) changes into

$$(2)\qquad \frac{d^2W}{dz^2}+\left\{-\frac{1}{4}+\frac{k}{z}+\frac{\frac{1}{4}-m^2}{z^2}\right\}W=0\ \ (m=l+\tfrac{1}{2}),$$

which is the confluent hypergeometric equation of Whittaker and Watson. One particular solution is given by the Kummer function $M_{k,m}(z)$. Since $2m$ is an integer, a second solution may be defined by a Neumann limit-form

$$N_{k,m}(z)=\lim_{\mu\to m}\frac{\Gamma(\frac{1}{2}+\mu+k)/\Gamma(\frac{1}{2}-\mu+k)\cdot I_{k,\mu}(z)\cos 2\mu\pi-I_{k,\mu}(z)}{\sin 2\mu\pi}$$

$$\left(I_{k,\mu}(z)=\frac{M_{k,\mu}(z)}{\Gamma(1+2\mu)}\right).$$

The author shows that $I_{k,m}(z)$ may be expanded in modified Bessel functions of the first kind:

$$I_{k,m}(z)=(\pi z)^{\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^r k\Gamma(k+\frac{1}{2}r)z^{r/2}}{r!\Gamma(k-\frac{1}{2}r+1)\Gamma(\frac{1}{2}+m+\frac{1}{2}r)}\,I_{m+\frac{1}{2}r}(\tfrac{1}{2}z)$$

and so he obtains an expression of the following form:

$$(3)\quad \pi N_{k,m}(z)=$$

$$\frac{\Gamma(\frac{1}{2}+m+k)}{\Gamma(\frac{1}{2}-m+k)}\{\tfrac{1}{2}\psi(\tfrac{1}{2}m+k)+\tfrac{1}{2}\psi(\tfrac{1}{2}-m+k)-\psi(2m)$$

$$-\psi(2m+1)+\log z\}I_{k,m}(z)-\sum_{r=0}^{\infty}(A_r z^{\frac{1}{2}r+m}+B_r z^{\frac{1}{2}r-m})I_{k,\frac{1}{2}r}(z),$$

where A_2 and B_2 are explicitly given in Γ-functions. Beyond the nucleus a well-behaved wave function for an atomic electron is given by the combination

$$W_{k,m}(z)=\Gamma(\tfrac{1}{2}+m+k)\cos\pi(k-m-\tfrac{1}{2})I_{k,m}(z)$$
$$+\Gamma(\tfrac{1}{2}-m+k)\sin\pi(k-m-\tfrac{1}{2})N_{k,m}(z).$$

In repulsive fields the solution is expressed by the functions

$$J_{k,m}(\zeta)=i^{-(m+k)}$$
$$Y_{k,m}(\zeta)=$$

$$\lim_{\mu\to m}\left\{\frac{\frac{1}{2}\pi\Gamma(\frac{1}{2}+\mu+ik)\Gamma(\frac{1}{2}+\mu-ik)(e^{\pi k}+e^{-\pi k}\cos 2\pi\mu)J_{k,\mu}(\zeta)}{\sin 2\pi\mu}\right.$$

$$\left.-\frac{J_{k,-\mu}(\zeta)}{\sin 2\pi\mu}\right\}.$$

analogous to (3) is obtained, with $J_{k,\frac{1}{2}r}(\zeta)$ in the place of $I_{k,\frac{1}{2}r}(z)$. When k and ζ are real, this expansion does not involve any imaginaries. The clearance of imaginaries is the main advantage gained by this expansion over power series expansion.

S. C. van Veen (Delft).

de Broglie, Louis. La signification du $|\Psi|^2$ pour les états stationnaires dans l'interprétation causale de la mécanique ondulatoire. C. R. Acad. Sci. Paris **243** (1956), 689–692.

Bohm and Vigier [Phys. Rev. (2) **96** (1954), 208–216; MR **16**, 654] have given a fluid model for the causal interpretation of quantum mechanics in which an arbitrary density distribution approaches $|\psi(x)|^2$ in the course of time. This approach to equilibrium arises as a result of Brownian motion. In the present note, the author applies this model to the case of a stationary state of an atom. There the causal interpretation predicts that an electron may stand still at a fixed position in apparent contradiction with the probability distribution $|\psi(x)|^2$. If, on the other hand, it is undergoing Brownian motion, it will be knocked from one position to another in such a manner that on the average its distribution is $|\psi(x)|^2$, although on the average it is at rest.

A. S. Wightman.

Van Hove, Léon. Von Neumann's contributions to quantum theory. Bull. Amer. Math. Soc. **64** (1958), 95–99.

The author reviews briefly the celebrated contributions of von Neumann to the mathematical framework of quantum theory, to quantum statistics and to the quantum theory of measurement. In particular, the spectral theorem for a hypermaximal hermitian operator, the uniqueness theorem for representations of the commutation relations, the theory of the statistical operator and the quantum ergodic theorem are mentioned.

A. S. Wightman (Princeton, N.J.).

Albertoni, S.; and Duimio, F. Hilbert spaces in quantum field theories. Nuovo Cimento (10) **6** (1957), 1193–1205.

This is a mainly expository article describing recent work on the theory of quantum mechanical systems with an infinite number of degrees of freedom. The infinite direct product of a denumerable number of Hilbert spaces is introduced, following von Neumann [Compositio Math. **6** (1938), 1–77], and the physical relevance of partial direct products ("bundles", in the notation of the authors) is described. The result of van Hove [Acad. Roy. Belg. Bull. Cl. Sci. (5) **37** (1951), 1055–1072; MR **14**, 117] and Miyatake [J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **2** (1952), 89–99; MR **14**, 705] is quoted, which states that the interaction hamiltonian of certain classes of field theories possesses only the trivial domain in the bundle of states with total number of particles finite. An example of Haag [Danske. Vid. Selsk. Mat.-Fys. Medd. **29** (1955), no. 12; MR **17**, 112], displaying the unitary inequivalence of certain representations of the commutation relations, is described in detail. The well known theory of a neutral scalar field interacting with external sources is worked out [L. van Hove, Physica **18** (1952), 145–159; MR **14**, 118]. In this connection the authors make a conjecture which, in the language of physicists, is that the annihilation and creation operators of the dressed particles should be unitary equivalent to those for the bare particles of the uncoupled system. This conjecture holds in the example just mentioned. The paper concludes with an appendix which contains a summary of parts of the paper by von Neumann quoted above. *A. S. Wightman.*

Klein, O. Some remarks on general relativity and the divergence problem of quantum field theory. Nuovo Cimento (10) **6** (1957), supplemento, 344–348.

It is argued that the length $[8\gamma h/c^3]^{\frac{1}{2}} \approx 10^{-32}$ cm (γ the gravitational constant, h Planck's constant, c the velocity of light) should provide a natural cut-off in quantum electrodynamics. Källén [Helv. Phys. Acta **25** (1952), 417–434; MR **14**, 435] and Lehmann [Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332] have shown that, in a theory invariant under the special theory of relativity, the singularities of the particle propagators are at least as bad in the theory of interacting fields as in the theory of free fields. It is argued that their result is not inconsistent with the above idea because the equation $i\dfrac{\partial\varphi}{\partial x^\mu} = [\varphi, P_\mu]$ is not valid in a theory invariant under the general theory of relativity. *A. S. Wightman* (Princeton, N.J.).

Dirac, P. A. M. The vacuum in quantum electrodynamics. Nuovo Cimento (10) **6** (1957), supplemento, 322–339.

The Hamiltonian of unrenormalized quantum electrodynamics is written in terms of longitudinal and transverse field variables and spinor field variables. These are, in turn, expressed in terms of annihilation and creation operators. An approximate expression for the vacuum state is sought in the form

$$e^{-i\epsilon(\bar{\psi},V\psi)}e^{-\bar{\psi}\lambda\psi}|S\rangle,$$

where $|S\rangle$ is the no-particle state of the annihilation and creation operators of the fields and V is an undetermined field operator describing the longitudinal field. (The first canonical transformation is due to Schwinger and is well-known; see, e.g., G. Källén, Encyclopedia of physics, vol. V, part 1, Springer, Berlin, 1958, pp. 324–327.) With a suitable choice of λ, one finds that H applied to this vector gives zero for the one-electron-pair amplitude. In conclusion, the author states that the following questions are still open: (a) is this state invariant under Lorentz transformations, (b) does the Hamiltonian have a minimum eigenvalue, and how well does this state approximate it? *A. S. Wightman* (Princeton, N.J.).

Heitler, W. The universal constant limiting the validity of the quantized field theories. Nuovo Cimento (10) **6** (1957), supplemento, 340–343.

A brief discussion of the idea that there exists a fundamental length which determines the range of validity of quantum field theories of mesons and nucleons. [See also W. Heisenberg, Ann. Physik (5) **32** (1938), 20–33; E. Arnous and W. Heitler, Nuovo Cimento (10) **2** (1955), 1282–1296; MR **17**, 810.] When estimated from the π^0–π^+ mass difference, this length is found to be about equal to the nucleon compton wave length. This, the author concludes, is consistent with our present knowledge. *A. S. Wightman* (Princeton, N.J.).

Feinberg, G. Selection rules implied by CP invariance. Phys. Rev. (2) **108** (1957), 878–881.

Strict invariance of a relativistic field theory under the product of the operations of charge conjugation and space inversion is assumed (but not necessarily under the operations separately) and some consequences for local interactions and S-matrix elements are obtained. For the case of a neutral spin zero field coupled to a spinor field a typical result is: the interaction may be of the form $\bar{\psi}\psi\phi$ or any combination of $\bar{\psi}\gamma_5\psi\phi$, $\bar{\psi}\gamma_\mu\psi\partial_\mu\phi$, $\bar{\psi}\gamma_\mu\gamma_5\psi\partial_\mu\phi$ but not both. A typical selection rule in this case is: a transition with $n_{ps}+n_{pv}+n_v+n_t+n(\varepsilon)$ odd is forbidden. Here n_{ps}, n_{pv}, etc., are the number of particles in the transition whose fields undergo ps, pv, etc., interactions respectively and $n(\varepsilon)$ is the number of times the totally antisymmetric tensor $\varepsilon_{\kappa\lambda\mu\nu}$ appears in the S matrix element.
A. S. Wightman (Princeton, N.J.).

Hack, M. N. Modified plane waves and rearrangement collisions. Phys. Rev. (2) **108** (1957), 1636–1641.

A derivation is presented of the familiar expression for the S matrix in terms of the perturbing potential and the appropriate solutions of the unperturbed problem. This analysis is extended to the S matrix for rearrangement collisions for which several equivalent forms are obtained.
H. Feshbach (Cambridge, Mass.).

Passatore, G. On polarization effects in Coulomb electron scattering. Nuovo Cimento (10) **6** (1957), 850–863.

The description of Coulomb scattering of relativistic electrons is expressed in terms of an operator which connects the Stokes parameters for the incident beam and the Stokes parameters for the scattered beam. The expression of this operator in terms of the scattering amplitudes is given, as well as the connection with the S matrix formalism. Both Born approximation and exact results are given, and the physical interpretation of their results discussed. *H. Feshbach* (Cambridge, Mass.).

Nakano, Tadao. A relativistic field theory of an extended particle. I. Progr. Theoret. Phys. **15** (1956), 333–368.

This is a fairly exhaustive study of a model particle, with an internal structure completely characterized by a single 3-vector. The special features of this model are that it can be described in a five-dimensional relativistic

formalism, and that it is consistent with a field equation of Bhabha's type for particles of arbitrary spin. This first paper does not consider interactions with other particles, but sets up, in turn, a classical non-relativistic theory, a quantized theory, a classical relativistic theory, a classical field theory, and a quantized field theory. The last is consistent with either Fermi or Bose statistics, according to whether the spin has a finite number of half-odd integral or integral eigenvalues. *H. S. Green* (Dublin).

Edwards, S. F. The range of validity of strong and weak coupling approaches in field theory. Proc. Roy. Soc. London. Ser. A. **243** (1958), 458–463.

Fixed source, charged, scalar meson theory is investigated, using Feynman's path integrals. The functional integral for the nucleon propagator is approximated to by expanding about a point of stationary phase. There are found to be two such points; and for weak coupling only one is important, for strong coupling, only the other. Thus the method leads to no smooth join between the weak and strong coupling extremes. Moreover, the weak coupling region contracts as the cut-off increases.
J. C. Taylor (London).

Bocchieri, P.; e Montaldi, E. Sull'operatore hamiltoniano delle teorie beta. Nuovo Cimento (10) **6** (1957), 710–718.

Different properties of the Hamiltonian of β-interactions are examined in the different subspaces of the total Hilbert space in which it is defined. Of these, only one is of physical interest. In none of the cases examined does the use of perturbation approximation seem justified.
M. Cini (Rome).

Filimonov, G. F.; and Shirokov, Iu. M. Multiple interaction Hamiltonians in quantum electrodynamics. Soviet Physics. JETP **5** (1957), 84–88.

This paper contains calculations similar to those of Schwinger [Phys. Rev. (2) **75** (1949), 651–679; MR **10**, 663]. By a succession of unitary transformations, the real processes of lower orders in the coupling constant are removed. The interaction Hamiltonian for the interaction of a pair of spinor particles in quantum electrodynamics is calculated. *C. A. Hurst* (Adelaide).

Imamura, Tsutomu. A new approach to a bound state. Progr. Theoret. Phys. **18** (1957), 559–566.

Two assumptions are made about the relation between the eigenvectors of the perturbed and unperturbed Hamiltonians of a scattering system. They are: (A) eigenvectors $\Psi_\mu(g)$ of the total Hamiltonian H converge weakly to eigenvectors Φ_μ of the free Hamiltonian H_0 as $g\to 0$; (B) the Hilbert space of the system can be divided into orthogonal subspaces which are invariant under H_0 and H, and are such that in each subspace, the bound state Ψ_B of H converges weakly to the lowest eigenstate Φ_0 of H_0. The author then defines a modified Hamiltonian H_0' which has the same eigenstates as H_0, but the eigenvalue of the lowest state is equal to the eigenvalue E_B of the bound state Ψ_B. The wave matrix derived from H_0', called the "extended wave matrix", transforms Φ_0 into Ψ_B if an additional supplementary condition is imposed. This supplementary condition determines the binding energy E_B. Only a subset of the scattering states of H_0 are transformed into scattering states of H. The wave matrix is isometric but not unitary. In contrast to the usual wave matrix, its range is the whole of Hilbert space, but its domain is a subspace.

Applications to the Lee model and to quantum electrodynamics are discussed. *C. A. Hurst* (Adelaide).

Schöpf, Hans-Georg. Über den Energie-Impulstensor Dirac-ähnlicher Felder. Ann. Physik (7) **1** (1958), 16–22.

In a previous paper [Ann. Physik (6) **18** (1956), 278–287; MR **19**, 98] the author derived the classical conservation laws for a lagrangian field theory. He here solves a system of partial differential equations to determine the form of lagrangian leading to field equations which are, in analogy with Dirac's, first order partial differential equations linear in the field variables. *A. J. Coleman.*

Omnes, R. A system of general relativistic equations of Low type. Nuovo Cimento (10) **5** (1957), 983–993.

Using Bogoliubov's methods, we give here a new demonstration of the Low equation, thus displaying its general character. The method has been extended to give a set of equations of Low type for the productions of mesons by mesons and one nucleon. *Author's summary.*

Kirzhnits, D. A. Contribution to field theory involving a cut-off factor. Soviet Physics. JETP **5** (1957), 445–451.

The author is interested in the uniqueness of the limit of a quantum field theory with a cut-off factor when the cut-off momentum goes to infinity. In particular, he discusses the arguments presented by Landau and collaborators [Dokl. Akad Nauk SSSR (N.S.) **95** (1954), 497–500, 773–776, 1177–1180; MR **16**, 315, 316; and later papers] about the consistency of quantum field theories. He accepts (at least for the purpose of the present discussion) the general integration technique of Landau et. al., and shows by explicit computations that, even so, the result of the argument depends very strongly on the detailed behaviour of the cut-off function in the neighbourhood of the cut-off momentum. Only if the cut-off function is sufficiently well behaved at high energies do the limits come out in accordance with the expectations expressed in the previous work. *G. Källén.*

Lehmann, H.; Symanzik, K.; and Zimmermann, W. On the formulation of quantized field theories. II. Nuovo Cimento (10) **6** (1957), 319–333.

This paper is a continuation of a previous paper by the same authors [Nuovo Cimento (10) **1** (1955), 205–225; MR **17**, 219], in which they discussed general properties of quantized fields. The main result of this new paper is the following. For any given theory (of a scalar field) in which there exist "incoming" and "outgoing" scattering states (and corresponding fields $A^{\text{in}}(x)$ and $A^{\text{out}}(x)$), and in which there also exists a unitary S-matrix $S=e^{i\eta}$, mapping the "out"-states onto the "in"-states, the authors define a "current"-operator $j(x)$ by

$$j(x)=-\int_0^1 d\lambda e^{-i\lambda\eta}\theta(x)e^{i\lambda\eta},$$

$$[A^{\text{in}}(x),\ \eta]=i\int\Delta(x-y)\theta(y)dy.$$

This current operator, which is so far defined only on the mass-shell (because of the factor $\Delta(x-y)$ in the second equation), is then extrapolated outside the mass-shell in an arbitrary way and used to define a "Heisenberg"-field $A(x)$ from

$$A(x)=A^{\text{in}}(x)-\int\Delta_R(x-y)j(y)dy.$$

This field $A(x)$ is not uniquely determined by this procedure and will, in general, not fulfil the "causality requirement" $[A(x), A(y)]=0$ for $x-y$ space-like. The restrictions on the S-matrix necessary to yield (at least) one causal field $A(x)$ are not discussed in detail.

The rest of the paper is mainly concerned with a discussion of the formal properties of iterated, retarded commutators of the field. Retarded commutators of this kind were first used by H. Umezawa and S. Kamefuchi [Progr. Theoret. Phys. **6** (1951), 543–558; MR **13**, 713].

G. Källén (Copenhagen).

Barashenkov, V. S. The construction of a phenomenological scattering matrix with non-local interaction. Soviet Physics. JETP **5** (1957), 313–315.

The author proposes a non-local theory of the S-matrix for which the nth order term $S_n=(-i)^n/n!\int_{-\infty}^{\infty}\rho^n\{H_1, \cdots, H_n\}d^4(x_1, \cdots, x_n)\ H_i=H(x_i)$ $=-ie\int\varphi^n(x_1)A(x_2)\phi(x_3)\sum_{i=1}^{3}F_i(x_1,x_2,x_3)\delta(x_i-x)\,d^4(x_1x_2x_3)$. F_i is a form factor and P^* is a modified chronological ordering defined by $P^*\{H_iH_j\}=P\{H_iH_j\}$ for timelike separations of x_i and x_j and $P^*\{H_iH_j\}=\frac{1}{2}[H_iH_j]_+$ for spacelike separations. It is claimed that the unitarity condition and the causality condition in macroscopic regions is thereby satisfied. *C. A. Hurst* (Adelaide).

Wong, David Y.; and Toll, John S. Causality and the dispersion relation: S-Matrix for the Maxwell field. Ann. Physics **1** (1957), 91–111.

This paper makes an important contribution to the problem of deriving dispersion relations for scattering of a classical Maxwell field by a fixed charged scatterer. The condition of causality (in the form that no signals can travel faster than the velocity of light) is imposed on the forward scattering of a sequence of primary wave packets φ^n, $n=0, 1, 2, \cdots$ in which the nth primary wave contains only partial waves for $l\geqq n$. Dispersion formulae are directly obtained for $\sin 2\delta_l$ for each l in terms of combinations of $\sin^2\delta_l$ plus the low energy limits of $e^{2i\delta_m}/\omega^{2m+1}$, $m\geqq l$. The results correspond to those one might obtain using the principle of micro-causality and neglecting creation thresholds. The importance of this paper lies, however, in showing how far one can go without using the principle of micro-causality in the field-theoretic sense.

A. Salam (London).

Bincer, Adam M. Scattering of longitudinally polarized fermions. Phys. Rev. (2) **107** (1957), 1434–1438.

The elastic differential cross section is computed for two longitudinally polarized Dirac particles which interact electromagnetically. The cross sections for initially parallel (ϕ_p) and initially antiparallel (ϕ_a) spins yield a ratio ϕ_p/ϕ_a which is a strong function of angle. It is therefore suggested that one use scattering as a method for measuring longitudinal polarization. Results are presented for Møller, Bhabha, and muon-electron scattering.

F. Rohrlich (Iowa City, Iowa).

Lipkin, H. J. Method for simplifying calculations involving products of operators in many-particle systems. Phys. Rev. (2) **108** (1957), 191–192.

This paper is concerned with the evaluation of sums of the general structure

$$S^{(n)}=\sum_{E_i\neq E_0}\frac{\langle 0|A|i\rangle\langle i|B|0\rangle+\langle 0|B|i\rangle\langle i|A|0\rangle}{(E_i-E_0)^n},$$

where $|0\rangle$ and $|i\rangle$ are two states of a quantum mechanical system, A and B are any two operators, E_i and E_0 are the eigenvalues of some operator H (e.g., the Hamiltonian of the system) for which $|0\rangle$ and $|i\rangle$ are eigenstates, and n is an integer. Sums of this kind appear, e.g., in the theory of rotating nuclei described with the aid of the so called "cranking model" [D. R. Inglis, Phys. Rev. (2) **96** (1954), 1059–1065]. The author writes $S^{(n)}=\sum_k S_k(\Delta E)^{-k}$, where S_k is defined to be a sum similar to the one shown in the formula for $S^{(n)}$, but without the denominator $(\Delta E)^n$, and summed only over states with a given value ΔE of the difference E_i-E_0. The convenience of this division of the summation lies in the fact that the quantum mechanical system studied in the applications which the author has in mind is highly degenerate. Therefore, even if there are many states $|i\rangle$ in the original sum, the sum over k contains only a small number of terms. The partial sums S_k are then computed from the following system of vacuum expectation values

$$\langle 0|[B, C_{2r+1}]|0\rangle=\sum_k(\Delta E_k)^{2r+1}S_k;$$
$$C_r=[H, C_{r-1}];\ C_1=[H, A].$$

If a sufficiently large number of these commutators can be easily computed (which is sometimes the case), the coefficients S_k can be determined from the resulting linear system of equations, and can afterwards be used to compute $S^{(n)}$. *G. Källén* (Copenhagen).

Abrikosov, A. A. The Compton effect at high energies. Soviet Physics. JETP **3** (1956), 474–483.

The Compton effect is investigated at high energies. Those terms are analyzed which arise from diagrams giving the maximum power of $\ln(\omega/m)$ with each power of e^2. Such terms are of order $[e^2\ln^2(\omega/m)]^n$. Formulas are given for single and multiple Compton effect, for the distribution of secondary photons in energy and angle, and for the energy distribution of secondary electrons. At small angles a peculiar change in the scattering is found, which leads to a change in the refractive index of energetic photons in matter. (Author's summary).

N. Rosen (Haifa).

Hall, D.; and Wightman, A. S. A theorem on invariant analytic functions with applications to relativistic quantum field theory. Mat.-Fys. Medd. Danske Vid. Selsk. **31** (1957), no. 5, 41 pp.

The paper consists of three parts. The first part gives a proof, with the aid of seven lemmas, of the following theorem [see Wightman,, Phys. Rev. **101** (1956), 860–866; MR **18**, 781]: An analytic function of n four-vector variables invariant under the orthrochronous Lorentz group is an analytic function of their scalar products. The second part is concerned with the domain of analyticity of the above theorem. This is the set of $n\times n$ complex matrices of the form $z_j\cdot z_k$ ($j, k=1, \cdots, n$), where $z_j=\xi_j-\eta_j$ is a complex vector, with ξ_j and η_j real, and η_j is in the future light cone. This set of z_j's is called the tube. It is then shown that an invariant function analytic in the tube has a boundary value which is an analytic function of $\xi_j\cdot\xi_k$ ($j, k=1, \cdots, n$) in a certain domain, and that the analytic function is uniquely determined once its values are known in certain subdomains of the boundary of the tube. The third part applies these results to the vacuum expectation values of a scalar field which is shown to be completely determined for all values of its argument by its values for space-like separations. In fact, for 2, 3, and 4 dimensions, it is completely determined by its values at isochronous points. It is further concluded that two

theories of an interacting field which satisfy canonical commutation relations must use different representations of the commutation relations in order to give different physical predictions. *M. J. Moravcsik.*

Hart, Robert W.; Gray, Ernest P.; and Guier, William H. Energy dependence of cross sections near threshold. One neutral and two charged reaction products. Phys. Rev. (2) **108** (1957), 1512–1522.

Wigner has shown that the energy-dependence of cross-section near the threshold energy at which a reaction barely occurs is independent of the details of the reaction process, and depends only on the number and charge of the products. The present work evaluates this dependence for the most difficult case so far worked out.

P. W. Anderson (Murray Hill, N.J.).

Davydov, A. S.; and Filippov, G. F. Moment of inertia of a system of interacting particles. Soviet Physics. JETP **5** (1957), 676–684.

This paper gives the dynamics associated with separating out the degrees of freedom of collective rotation of a system of particles, and demonstrates the assumptions necessary if a moment of inertia is to be defined. No attempt is made to give a microscopic theory of when these assumptions are valid and what the value of the moment is. *P. W. Anderson* (Murray Hill, N.J.).

Temkin, Aaron. Polarization and exchange effects in the scattering of electrons from atoms with application to oxygen. Phys. Rev. (2) **107** (1957), 1004–1012.

A method is developed to include polarization effects in low-energy scattering of electrons from atoms, within the Hartree-Fock formalism. This description is different from the usual expansion in terms of the eigenfunctions of the atom. The method associates with each atomic orbital perturbed or polarized parts which also depend parametricaly on the coordinates of the scattered particle. [From the author's summary.] *P. M. Morse.*

Nesbet, R. K. Brueckner's theory and the method of superposition of configurations. Phys. Rev. (2) **109** (1958), 1632–1638.

The method of superposition of configurations, which provides a general solution for the many-particle problems in quantum mechanics, is reformulated, insofar as possible, in terms of an effective two-particle operator, so that equations may be compared with those used in the Brueckner theory. The two-particle operator depends upon a choice of basis for its matrix representation; two are considered in this paper; the first is a condition equivalent to the Hartree-Fock equations; the second, a condition which corresponds closely to the Brueckner method. A formalism based on the variational principle is developed which is as nearly as possible equivalent to the Brueckner method, but resolves many of its ad hoc features.

Important differences may occur between the two methods. In some cases non-zero single-particle matrix elements occur which cannot be represented by the Brueckner formalism based on two-particle operators, nor can equations be written in terms of products of operators, a formalism assumed in the Brueckner theory. Also, it is shown that the Brueckner condition is not equivalent to the condition which minimizes the energy.

The discussion is limited to systems with a finite number of particles. *C. Froese* (Vancouver, B.C.).

Nesbet, R. K. The use of projection operators in the configuration interaction problem. Ann. Physics **3** (1958), 397–407.

The straightforward evaluation of the configuration interaction matrix for a many-particle Hamiltonian in a basis of single determinants leads to matrix elements which are quadratic forms in the coefficients of the determinantal expansions. In this paper, a projection operator is introduced which, when applied to a single determinant, forms a linear combination of determinantal functions proportional to a totally antisymmetric many-particle function with definite transformational properties. The matrix elements between any two such functions can be expressed as linear forms in the coefficients of the determinantal expansion; the constant of proportionality need not be determined. A method is given for constructing projected determinants from an orthonormal set of totally antisymmetric functions with the same transformation properties under a symmetry group for which the Hamiltonian remains invariant. Equations are stated which determine transformation coefficients that completely reduce the representation of an arbitrary symmetry group.

The work is intended to complement the general method of Boys and is to be used in connection with the Hartree-Fock method. The method is applied to the two ^2D terms of the d^3 configuration as well as the ^2D term of the s^2d configuration. *C. Froese* (Vancouver, B.C.).

Rose, M. E. Statistical tensors for oriented nuclei. Phys. Rev. (2) **108** (1957), 362–365.

The statistical tensor G_ν ($\nu=0, 1, \cdots$) which determines the angular distribution of radiation from oriented nuclei is the average over the spin population of the component $T_{\nu 0}$ of an irreducible tensor operator of rank ν. Explicit expressions for $T_{\nu 0}$ ($\nu=0$ to 4) are presented, and the relation of $T_{\nu 0}$ with the multipole operators in spin space is derived. *F. Rohrlich.*

★ **Kauzmann, Walter. Quantum chemistry: An introduction.** Academic Press Inc., Publishers, New York, 1957. xii+744 pp. $12.00.

This book should prove very useful to those embarking on a study of quantum chemistry with a limited mathematical background. A special effort has been made to separate the mathematical difficulties of the subject from the difficulties associated with the new physical concepts of quantum mechanics. This is achieved by including an introductory section which develops the mathematical theory of the vibrations of classical systems. In this section many of the concepts and tools of quantum theory, including perturbation theory, the variational method, non-commuting Hermitian operations, resonance and the hybridization of wave functions are introduced in a purely classical context. Again at a later stage the classical theory of the interaction of light and matter is developed at some length and it is shown that many of the results of this theory may be taken over into the quantum theory by reinterpreting the significance of certain parameters in the equations. This technique is particularly successful in the discussion of optical rotatory power, a field to which the author has made important contributions, and brings out clearly the physical basis for this phenomenon, which is often masked by a purely deductive quantum mechanical treatment.

The central sections of the book are devoted to a lucid exposition of the Schrödinger formulation of quantum

mechanics and the solution of the stationary state wave equation for simple atomic and molecular systems. Ab initio calculations of the ground states of the helium atom, the hydrogen molecule ion and the hydrogen molecule, and of the spectrum of benzene are discussed at considerable length. This detailed analysis of simple systems should prove very valuable to the student, although the section on helium-like ions is marred by an unusual and confusing definition of the correlation energy (p. 287). The great difficulties encountered in extending such complete calculations to more complex molecules are clearly brought out, as are the uses and abuses of simpler approximate and empirical calculations. It is pointed out that, to date, the principal service of quantum theory to chemistry has not been that of a tool which dispenses with the need for experimental measurements, but rather that of a new language or framework withni which different types of chemical and physical measurements may be embedded and related. This section includes a discussion of nuclear quadrupole coupling and the chemical shift in nuclear magnetic resonance, subjects not previously dealt with in a text on quantum chemistry.

The printing and indexing are good. There are few obvious misprints. As well as the usual table of contents and subject index there is an index of mathematical symbols and a useful table giving suggested plans of study for various topics. *A. C. Hurley* (Melbourne).

Odiot, Simone. Etude des corrélations entre les positions des électrons. Cahiers de Phys. **11** (1957), 181–202.

A discussion is given of the correlation of the positions of electrons in atoms and molecules. A distinction is drawn between correlations arising from the antisymmetry of the total wave function and those due to electrostatic repulsion. The effect of these correlations on observable quantities is investigated, with special reference to electron capture by the atomic nucleus. *A. C. Hurley.*

Maes, Serge; et Amat, Gilbert. Sur les calculs de perturbation effectués par la méthode de Van Vleck. Cahiers de Phys. **11** (1957), 277–284.

Van Vleck [Phys. Rev. (2) **33** (1929), 467–506] has introduced a method for simplifying perturbation calculations by means of a unitary contact transformation. It is shown that this transformation is not uniquely determined, but that this has no effect on the final results, which coincide with those of the classical perturbation theory. The method is applied to the vibrational and rotational motion of a polyatomic molecule.

A. C. Hurley (Melbourne).

Young, James E. Perturbation expansions in the formal theory of scattering. Phys. Rev. (2) **109** (1958), 2141–2144.

This formal paper is a re-examination of previous work on the scattering of quantum-mechanical systems by two potentials. The approximations underlying the results of Gell-Mann and Goldberger [Phys. Rev. (2) **91** (1953), 398–408; MR **15**, 382] are discussed and the relationship between their work and that of Watson [ibid. **88** (1952), 1163–1171] is explained. The general picture used here is that of a perturbation by a so-called surface potential on the classical Wigner reaction model. Finally the general case of distinct pairs of initial and final state interactions is considered, exact formal expressions are written down, and various approximations are discussed as a function of the interaction radius. *M. J. Moravcsik.*

Elliott, J. P. Collective motion in the nuclear shell model. I. Classification schemes for states of mixed configurations. Proc. Roy. Soc. London. Ser. A. **245** (1958), 128–145.

The construction of wave functions for the bound states of atomic nuclei has developed along two major lines which in many respects appear to be quite distinct. The shell model proceeds by use of wave functions for individual particles in suitable potential fields, while the collective model attempts to start from a description of the correlated motions of all of the particles. Each of these procedures has been distinctly successful in limited applications. The author starts from the view that the two procedures provide different sets of basis states in terms of which the true states can be found by taking suitable linear combinations. A major problem is then to establish the connection between the two extreme models by finding the transformation scheme between their bases. The present paper is the first of a series in which this program is to be developed. The method used is a straightforward classification of the degeneracies of the configurations given in the shell model, the associated wave functions being then combined by linear transformations into sets which have properties resembling collective motions. The classification and re-combination of the wave functions is carried out by group theoretical methods developed earlier by G. Racah [Phys. Rev. (2) **76** (1949), 1352–1365]. *E. L. Hill.*

Kikuta, Takashi. Extensions of variational methods. I. Super-stationary variational method. Progr. Theoret. Phys. **14** (1955), 457–472.

By selecting a set of trial functions, the author gets several methods for improving the degree of approximation for the stationary character in variational methods. He means by "super stationary" variational methods that the first, second and third variations of the expressions under consideration vanish for any infinitesimal variations of trial functions. The examples to which this method is applied are: (1) To find the eigenvalue in the eigenvalue problems with discrete spectra only (e.g., to find the potential depth in the deuteron problem). (2) To find the discrete eigenvalues for the problems with both discrete and continuous spectra (e.g., to find the energy level of the deuteron). (3) To solve problems for continuous spectra by integral equations. (4) To solve problems for continuous spectra by differential equations. Also, a method is mentioned for improving approximation of the stationary character in a successive manner in each case, and it is shown that a simple method for this improvement exists in cases (1) and (2). *M. Pinl* (Cologne).

Kikuta, Takashi. Extensions of variational methods. II. Two parameter eigenvalue problem for the deuteron state. Progr. Theoret. Phys. **15** (1956), 50–62.

The deuteron problem under consideration concerns the treatment of the determination of the depths of central and tensor potentials to get the correct experimental values for the binding energy γ^2 and the electric quadrupole moment Q, assuming a suitable function for the potential. The author considers these two depth values as "two-parameter eigenvalues", using the following approximation relation among Q, γ^2 and the mixing parameter ε of the D-state to the S-state: $\tan \varepsilon = -\sqrt{2}Q\gamma^2$; $\hbar^2\gamma^2$=(binding energy of deuteron)×(nucleon mass). The author can determine both depths simultaneously without recourse to approximate methods such as

trial-and-error. He generalizes, furthermore, the variation principle corresponding to the Rayleigh-Ritz variational method to a many-parameter eigenvalue problem.

M. Pinl (Cologne).

Ladányi, K. Zur statistischen Näherung des wellenmechanischen Mehrteilchenproblems. II. Acta Phys. Acad. Sci. Hungar. **7** (1957), 267–272.

In place of the single-particle wave functions used in the previous work [same Acta **7** (1957), 161–166; MR **19**, 612], a set of orthogonal functions of a different form is introduced. It is found that the Hartree-Fock expression for the energy of the ground state is nearly equal to that given by the statistical model, even for a relatively small number of electrons. The non-stationary problem is discussed. *N. Rosen* (Haifa).

Borowitz, Sidney; and Klein, Milton M. Perturbation calculation of the inelastic scattering of electrons by hydrogen atoms. Phys. Rev. (2) **103** (1956), 612–619.

Louck, J. D. New recursion relation for the Clebsch-Gordan coefficients. Phys. Rev. (2) **110** (1958), 815–816.

A recursion relation for the Clebsch-Gordan coefficients is presented which will facilitate the construction of further tables. *Author's summary.*

Hartree, D. R. Variation of atomic wave functions with atomic number. Rev. Mod. Phys. **30** (1958), 63–68.

Reprinted from the proceedings of the international conference in crystal physics. See MR **19**, 101.

Mazo, R. M.; and Zemach, A. C. Diffraction of neutrons by imperfect gases. Phys. Rev. (2) **109** (1958), 1564–1572.

A compact symbolic method expressing a many particle function (the expectation value of an operator with respect to a many body wave function) in terms of cluster functions is developed. This is applied to obtain the density expansion for the thermal average of an arbitrary operator function in terms of irreducible cluster integrals. The classical high temperature limiting value is obtained with the aid of the Baker-Hausdorff theorem. These methods are applied to the discussion of elastic and inelastic scattering of neutrons by unperfect gases. It is found that inelastic effects are much less important than the interference cross-section. *H. Feshbach.*

Phan-Van-Loc. Une nouvelle manière d'établir l'expression mathématique du principe de Huygens en théorie de l'électron de Dirac. C. R. Acad. Sci. Paris **246** (1958), 388–390.

A derivation of the Green's function expression for the value of wave function for the Dirac equation in terms of its values on a surrounding surface. *H. Feshbach.*

Feld, Bernard T. Kinematics of β decay and parity nonconservation in weak interactions. Phys. Rev. (2) **107** (1957), 797–804.

The kinematics of the β decay process for allowed transitions is discussed with the aid of the decomposition of the electron and neutrino wave functions into large and small components, each of which can be separately described by its angular momentum and parity. By adding to these wave functions wave functions of opposite parity, it becomes possible to describe the effects of parity violating interactions. Various experimental situations are discussed: the angular distribution of electrons emitted from oriented nucleii and from polarized μ mesons, and the angular distribution of π mesons from polarized spin 1 K mesons. *H. Feshbach.*

Horwitz, L. P. Nonlinear coupling in low-energy meson theory. Phys. Rev. (2) **108** (1957), 886–890.

It is shown that if the linear pseudovector meson-nucleon coupling in the fixed extended-source meson theory is replaced by a much more general non-linear coupling, the scattering equations of the Chew-Low-Wick formalism remain unchanged within the one-meson intermediate-state approximation. *S. N. Gupta.*

Sokolov, A. A. On the relativistic motion of electrons in a magnetic field with account taken of quantum effects. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **11** (1956), no. 2, 27–46. (Russian)

Various relations are derived for the intensity of radiation and the effect of radiation reaction in the case of the motion of high-energy electrons in a magnetic field according to quantum mechanics. Most of the material of this article has appeared elsewhere in English [Nuovo Cimento (10) **4** (1956), supplemento 743–759; MR **19**, 217]. *N. Rosen* (Haifa).

Aizu, Ko. General theory of particle mixtures. Nuovo Cimento (10) **6** (1957), 1040–1051.

The Weisskopf-Wigner theory of decay rates [Z. Physik. **63** (1930), 54–73] is extended to include situations where the initial state is degenerate. A general treatment of this problem is given including the possibility of mass shifts arising. Examples of this situation are the $K^0{}_1-K^0{}_2$ decays and nuclear β and γ decays where several modes are available. In the latter case it is shown that a selection rule (conservation of angular momentum) forces the system to have only one lifetime, which is not the case in K^0 decays. If strong interactions were to violate charge conjugation invariance and hence produce a mass shift Δ between the K^0 and $\bar{K}^0$, it is shown that $\Delta \leqq 10^{-5}$ ev in order that the K^0 system maintain two distinct lifetimes. Several other examples are considered.

R. Arnowitt (Syracuse, N.Y.).

Bludman, Sidney A. Some theoretical consequences of a particle having mass zero. Phys. Rev. (2) **107** (1957), 1163–1168.

The scalar and vector meson equations in the Duffin-Kemmer formalism are written in Hamiltonian form for non-zero mass, and in a Hamiltonian-like formalism for zero mass: $\partial_t(\gamma\psi)=\underset{\sim}{\alpha}\cdot\underset{\sim}{p}(\gamma\psi)$, where γ is the mass operator, satisfying $\gamma^3=\gamma$, and $\underset{\sim}{\alpha}=i(\beta_4\underset{\sim}{\beta}-\underset{\sim}{\beta}\beta_4)$. In this latter case, the supplementary condition $[\underset{\sim}{\beta}\cdot\underset{\sim}{\partial}\beta_4{}^2+(1-\beta_4{}^2)\gamma]\psi=0$ is also imposed. The element β_5 is then used to split the field into two sets of three components which are invariant under proper Lorentz transformations {The projection operator $\frac{1}{2}(1\pm\beta_5)$ of equation (28) should be $\frac{1}{2}(1\pm\beta_5)\beta_5$}. The general case for arbitrary spin is stated, namely: any representation $\psi(0, s)$ of the inhomogeneous improper Lorentz group splits into two representations $\psi_\pm(0, s)$ which are inequivalent except under space reflections. The massless Dirac equation is shown to be invariant under the proper and improper conformal groups.

C. A. Hurst (Adelaide).

Lenard, Andrew. Spin reversal in scattering processes. Phys. Rev. (2) **107** (1957), 1712–1713.

The effect on the scattering matrix element of reversing the spins of all particles is discussed. An exact relation is shown to be the consequence of invariance under combined space and time inversion and the unitarity of the S matrix. In lowest order perturbation theory this gives a symmetry property of the matrix element. The implication for particles in arbitrary states of polarization is pointed out. (Author's summary.)

F. Rohrlich.

Matthews, P. T. The parity of elementary particles. Nuovo Cimento (10) **6** (1957), 642–649.

The author discusses the parity of the elementary particles, using gauge invariances of the first kind to generate selection rules and considering charge independence. The three gauge groups considered are those associated with baryon number, charge, and isofermion number. The possibility that parity is not conserved in the weak decays (now known to occur for the Λ^0 meson) is taken into account. This leads, for example, to the remarkable situation that the relative parity of Λ^0 and n is not measurable because the reaction which could be used to compare their parities, $\Lambda^0 \to n+\pi^0$, does not conserve parity. The full discussion of the intrinsic parities of the presently known elementary particles is too complicated to summarize here.

A. S. Wightman.

Schwinger, Julian. A theory of the fundamental interactions. Ann. Physics **2** (1957), 407–434.

This is a "connected series of speculations" about elementary particles and their interactions in which, among the pions and baryons at least, a four-dimensional isotopic space symmetry may sometimes be revealed. One suggestion made is that a highly unstable, and therefore unobserved, iso-scalar, scalar meson might exist and allow an interlacing of isotopic and γ_5-transformations (provided that certain bare masses, but not the corresponding observed masses, were zero). K-mesons break the four-dimensional symmetry in a way connected with the different parities assigned to nucleons and cascade particles.

Leptons are included in the isotopic scheme; the muon's mass is ascribed to an interaction with the scalar meson. The photon becomes part of an iso-vector, its sisters being charged vector mesons which are heavy (apparantly about 30 nucleon masses) and are intermediaries in weak interactions. The neutrino is a four component spinor, but parity non-conservation is stated to appear naturally in the scheme.

J. C. Taylor.

Średniawa, Bronisław. Sur une modification de Lorentz-invariant formalisme dans la théorie de fusion. Acta Phys. Polon. **16** (1957), 399–405. (Russian summary)

A modification is given of de Broglie's fusion theory of spins, for the case of two fermions of spin $\frac{1}{2}$. Each can be either a Majorana or a Dirac particle. It is claimed that calculations are simplified in the proposed formalism.

H. W. Lewis (Madison, Wis.).

See also: Statistical Thermodynamics and Mechanics: Gurzhi. **Structure of Matter:** Van Kranendonk and Van Vleck. **Fluid Mechanics, Acoustics:** v. Krzywoblocki and Whittenbury. **Relativity:** Bakke and Wergeland.

Relativity

Słowikowski, W. On a certain variation problem in P-geometry and its relations to physical problems. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 385–386, XXXI. (Russian summary)

This is a continuation of a paper by the same author on applications of the Pauli ring [same Bull. **4** (1956), 313–320; MR **18**, 447]. Equations similar to those of Dirac and Proca are derived from a variational principle.

A. Trautman (Warsaw).

Fikhtengol'ts, I. G. Application of the Dirac-Fock-Podol'skii method to a mechanical many-body problem. Soviet Physics. JETP **5** (1957), 1144–1149.

A formulation is given for the integrals of the motion corresponding to the mechanical degrees of freedom of a system of particles in relativistic motion. The particles are expressed in terms of space-time coordinates, as in the "many-time" formalism. The first integrals are deduced by consideration of the effect of the infinitesimal generators of the Lorentz group of transformations on the Lagrangian function of the system. *E. L. Hill.*

Bakke, F.; and Wergeland, H. Double-valued solutions to the relativistic Kepler problem. I. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 99–105.

The authors find a solution of the Dirac equation of the form:

$$\Psi_k = N_k e^{-\gamma_{12}\varphi/2}\Theta_k(\vartheta)R_K(n)\Gamma,$$

where N is a normalization, φ and ϑ are the spherical coordinates, while γ_{12} is the commutator of the Dirac operators γ_1, and γ_2. Γ is $\frac{1}{2}(1+\gamma_4)(i\varepsilon-\gamma_3)$ where ε is $(E+mc^2)/cp$. The function Θ_k is $(\sin\vartheta)^{-\frac{1}{2}}\exp(ik+\gamma_{12}/2)\vartheta$. R is a radial function proportional to the usual confluent hypergeometric function. Finally, k is a parameter which turns out to be quantized and related to the total angular momentum. Although this wave function gives the correct energy levels and values of the total angular momentum it does not reproduce the correct values of the z component of the angular momentum. The authors do not explain the reason for this although they imply that it is connected with the multivalued nature of Ψ in its dependence on φ. *H. Feshbach* (Cambridge, Mass.).

Chernikov, N. A. A generalized problem on the stochastic motion of a particle. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 1030–1032. (Russian)

The problem of particles having a finite lifetime and the property of being able to be created (with subsequent annihilation) any number of times is treated relativistically in the seven-dimensional space of the coordinates and velocity components of a particle. A kinetic equation is derived. *N. Rosen* (Haifa).

Bergmann, Otto. Einige Bemerkungen zum Uhrenparadox. Acta Phys. Austriaca **11** (1957), 377–389.

Following a brief treatment of the clock paradox in Special Relativity, the author examines infinitely small oscillations in a system of geodesic coordinates within the framework of General Relativity. He calculates the time-difference for the interval between successive coincidences of two clocks, one of which remains at rest at the origin, while the other makes these successive oscillations. He then considers the same problem for an electrically charged clock, assuming the presence of a

uniform magnetic field in addition to the gravitational field. Finally, he briefly examines a new definition of an ideal electrically charged clock. *G. J. Whitrow.*

Arcidiacono, Giuseppe. La elettrodinamica e la idrodinamica nella "teoria di relatività finale". Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 616–622.

In a previous paper [same Rend. (8) **18** (1955), 515–519; MR **17**, 1244], the author obtained the generalized Maxwell equations in the theory of "final relativity", that is, the relativistic theory invariant under a "final" group, of which the Lorentz group is a limiting case when a certain parameter $R\to\infty$, and which cannot be itself regarded as the limiting case of another group of the same type. He showed that these Maxwell equations reduce to two groups when $R\to\infty$ (that is, in special relativity), the first group being the well-known classical equations of electrodynamics and the second group a new set of equations which, in vacuo, are of the form rot $\mathbf{G}=\mathbf{h}$, grad $G_0+\partial\mathbf{G}/c\partial t=\mathbf{k}$, $\partial G_0/c\partial t+\operatorname{div}\mathbf{G}=0$, where $\mathbf{h}$ and $\mathbf{k}$ are sources of the field. In the present paper, these new equations are interpretated as the relativistic hydrodynamical equations of a perfect incompressible fluid.

A. J. McConnell (Dublin).

Pham Mau Quan. Inductions électromagnétiques dans un milieu anisotrope relativiste. C. R. Acad. Sci. Paris **245** (1957), 1782–1785.

Lampariello, Giovanni. Le oscillazioni di un plasma sulla base del principio einsteiniano dell'impulso. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 477–482.

Pham Tan Hoang. Comparaison entre la méthode du tenseur d'impulsion-énergie et la méthode des singularités. C. R. Acad. Sci. Paris **246** (1958), 1497–1500.

On the assumption that the metric is euclidean at infinity and may be expanded in powers of c^{-2}, it is shown that the equations of motion of n bodies in general relativity are equivalent to the vanishing of certain integrals, involving the curvature and energy-momentum tensors, taken over the surfaces of the bodies.

A. J. Coleman (Toronto, Ont.).

Taub, A. H. Approximate solutions of the Einstein equations for isentropic motions of plane-symmetric distributions of perfect fluids. Phys. Rev. (2) **107** (1957), 884–900.

The author introduces a new approximation procedure for the solution of Einstein's field equations of general relativity. This method is discussed in considerable detail by means of the solution of the field equations for a plane-symmetric space-time, where the stress-energy tensor describes a perfect fluid with an arbitrary caloric equation of state in isentropic motion. According to this method, the special theory of relativity provides the zero-order approximation to the solution, and thus the first part of the paper is devoted to a discussion of the formulation and solution of the 1-dimensional motion of a perfect fluid. This is done by means of orthogonal comoving coordinates, which are introduced here in a most ingenious manner. This gives rise to a discussion of special relativistic hydrodynamics, and methods for solving specific problems are devised.

In his approach to the general theory, the author makes use of a co-moving coordinate system in a space-time with plane-symmetry, in which the gravitational field is created by a perfect fluid at constant entropy, whose line element had been previously determined [Phys. Rev. (2) **103** (1956), 454–467; MR **18**, 177]. The approximation procedure for solving the field equations is similar to the one used by Einstein and Infeld [Canad. J. Math. **1** (1949), 209–241; MR **11**, 59], but it differs from the latter scheme in an essential aspect, namely, that expansions of the coefficients of the metric tensor in terms of the constant $k=8\pi G/c^2$ (G=Newton's constant of gravitation) are used instead of some velocity. It is, therefore, not necessary to assume that the time variations of the field quantities are small compared to the spatial ones. This seems to be the crux of the author's method. A sequence of sets of equations are thus obtained, each set arising from the coefficient of a single power of k. The question concerning the existence of shock waves in general relativity is raised; this is suggested by the appearance of shock waves in special relativity, and it is indicated that this may require a suitable generalisation of the general theory of relativity. The paper concludes with some relevant remarks concerning gravitational waves. It is, unfortunately, impossible to give an adequate description of the author's results and methods within the bounds of a short review.

H. Rund (Durban).

Newman, Ezra; and Bergmann, Peter G. Observables in singular theories by systematic approximation. Rev. Mod. Phys. **29** (1957), 443–449.

By "singular theories" the authors mean theories which contain constraint equations arising due to the existence of invariance groups which depend upon functions (such as the gauge group of electrodynamics and the coordinate transformation group of general relativity). The difficulty in such theories lies in the fact that there exists an arbitrariness arising due to the function group and hence, not all variables are truly observables. In general relativity, the situation is further complicated by the high degree of non-linearity of the theory. The purpose of the paper is to construct, in a perturbation power series, the true observables for such classical non-linear theories. The generalized coordinates are expanded in a power series of which the zero'th order term is a constant (such an approach is appropriate for general relativity since the zero'th order field is the Lorentz metric). The corresponding Euler equations and the function group invariance for each order of approximation are determined. The dynamical equations of each order is linear in the variables of that order and inhomogeneous in the variables of lower order. The analysis of finding the Poisson bracket relations for each order then follows along the lines indicated by Dirac [Canad. J. Math. **2** (1950), 129–148; **3** (1951), 1–23; MR **13**, 306] and an explicit prescription for constructing the observables of a given order is written down. Care is taken to show that the perturbation procedure doesn't change the invariance properties of the theory or introduce extra constraints. *R. Arnowitt.*

Joseph, V. Physical properties of some empty space-times. Proc. Cambridge Philos. Soc. **53** (1957), 836–842.

This paper is concerned with the problem of the regularity of solutions of Einstein's field equations in empty space $R_{\mu\nu}=0$. A solution is called regular if the $g_{\mu\nu}$ are bounded and $\det g_{\mu\nu}\neq 0$ for all values of the coordinates; in the contrary case it is called irregular. The regularity

or otherwise of a solution is evidently dependent upon the choice of coordinate system, and the question arises as to the distinction between "physical irregularity" and "coordinate irregularity".

Certain metrics of A. H. Taub [Ann. of Math. (2) **53** (1951), 472–490; MR **12**, 865] are examined on the basis of the canonical forms of the curvature tensor introduced by A. Z. Petrov [Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8, 55–69; MR **17**, 892]. These can be constructed with the aid of sets of four suitable mutually orthogonal unit vectors. The author concludes that none of the solutions represent non-flat space-times, everywhere regular and of infinite extent. The implications of this as regards Mach's Principle are commented upon. *H. A. Buchdahl.*

Synge, J. L. An invariant gravitational density. Proc. Roy. Irish Acad. Sect. A. **58** (1957), 29–39.

In the general theory of relativity gravitational energy is represented by a pseudo-tensor t_k^r whose components can be made to vanish at any arbitrarily chosen event, or along any given curve in space-time by a suitable choice of coordinates. The author of the present paper therefore considers the physical meaning of t_k^r to be obscure, and he introduces a single invariant $H(\Lambda)$, which he calls "gravitational density", its integral over a four-dimensional domain being called the "quantity of gravity" within it. $H(\Lambda)$ is constructed from (i) the components R_{abcd} of the curvature tensor, (ii) an arbitrarily selected timelike unit vector Λ^r, and (iii) a positive weighting function W. $H(\Lambda)$ is positive definite and if it vanishes at a point E then $R_{abcd}=0$, i.e. space-time is flat at E. The Schwarzschild field is considered by way of example. ($H(\Lambda)$ depends upon the largely arbitrary choice of Λ^r and of W, and any particular $H(\Lambda)$ does not appear to have any clear physical meaning. The author comments upon this fact.)

H. A. Buchdahl (Hobart).

Verma, Daya Nand; and Roy, Swadesh Ranjan. Special metric forms and their gravitational significance. Bull. Calcutta Math. Soc. **48** (1956), 129–137.

The problem is to decide whether a simple non-trivial solution exists for the field equations of general relativity theory when the line-element contains only one variable potential. The cases when the square of the line-element is

(i) $-\varphi dx^2-dy^2-dz^2+dt^2$, (ii) $-dx^2-dy^2-dz^2+\varphi dt^2$,

(iii) $-\varphi(dx^2+dy^2+dz^2)+dt^2$, (iv) $-\varphi(dx^2+dy^2+dz^2-dt^2)$;

are discussed in detail, and a general solution is obtained in each case for the function φ, when matter is assumed to be distributed as a perfect gas at rest relative to the frame of reference. *A. J. McConnell* (Dublin).

Singh, K. P. Projective curvature tensor and relativistic gravitation. Bull. Calcutta Math. Soc. **48** (1956), 153–154.

The author investigates the effect of the vanishing of the projective curvature tensor of a Riemannian V_N and of its divergence on the gravitational field associated with the V_N. He finds that in the first instance there can be no gravitational field, and in the second instance there cannot be a perfect fluid distribution of matter at rest relative to the frame of reference. *A. J. McConnell.*

Pirani, F. A. E. Tetrad formulation of general relativity theory. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 143–146, XIII–XIV. (Russian summary)

Many difficulties of interpretation in general relativity theory arise from the use of coordinate systems to which no exact physical significance is ascribed. This may be avoided by introducing at every space-time point an orthonormal tetrad, of which one is interpretated as the 4-velocity of an observer at the point, the other three being interpretated as the local Cartesian space axes used by the observer. The theory can then be formulated in terms of the physical components of vectors and tensors, that is, the local components referred to the tetrad.

A. J. McConnell (Dublin).

Infeld, Leopold. Equations of motion in general relativity theory and the action principle. Rev. Mod. Phys. **29** (1957), 398–411.

This paper is identical to that in Acta Phys. Polon. **16** (1957), 177–210 [MR **19**, 815].

Pham Tan Hoang. Sur la méthode des singularités en relativité générale. C. R. Acad. Sci. Paris **246** (1958), 61–64.

The author describes the EIH method [see the paper listed above and previous papers cited there]. He expands in power series not the usual covariant tensor $g_{\mu\nu}$, but the contravariant tensor density $g^{\mu\nu}$, which he contends is more convenient. In the second and third approximation orders he deduces the constancy of mass and the Newtonian equations of motion [as in the usual theory].

F. A. E. Pirani (Baltimore, Md.).

Brahmachary, R. L. A class of exact solutions of the combined gravitational and electro-magnetic field equations of general relativity. Nuovo Cimento (10) **6** (1957), 1502–1506.

A solution of Einstein's equations for a spherically symmetric field as given by A. S. Eddington [The mathematical theory of relativity, 2nd ed., Cambridge, 1924, p. 185], is obtained corresponding to a finite electron surrounded by matter. The pressure is a constant, and the density is equal to a constant less a term varying inversely as the fourth power of the radial coordinate r. The solution is defined for a region which imposes a lower limit on r which is set equal to 10^{-13} cm. The case $p=0$ leads also to an upper limit for r. *A. J. Coleman.*

Pachner, Jaroslav. Über die Kompatibilität der Feldgleichungen, Erhaltungssätze und Bewegungsgleichungen in der unitären Feldtheorie. Ann. Physik (6) **20** (1957), 368–380.

The present paper is a continuation of the author's recent approach [Ann. Physik (6) **19** (1957), 353–368; MR **19**, 615] to the problem of devising a "unified" field theory, in which he postulates a new Hamiltonian. The latter is compared in detail with the Hamiltonians of earlier field theories, which are subjected to a critical survey. It is shown that the field equations which had been derived in the above-mentioned article are compatible, this analysis being based on results due to Hlavatý and Sáenz [J. Rational Mech. Anal. **2** (1953), 523–536; MR **14**, 1132]. There are 18 equations for the 16 component of the (non-symmetric) metric tensor $g_{\lambda\mu}$, but there are also 6 identities, so that only 12 equations are independent, as required by the general theory. Of these 6 identities, 4 are interpreted as conservation theorems for momentum and energy, the remaining 2 supposedly representing the conservation of electric and magnetic current density respectively. Further implications of certain peculiarities of the author's earlier spherically

symmetric static solution are discussed. The equations of motion of a test particle are derived from the usual variational principle (which involves a single integral along a world-line). Explicit approximations for the motion of a particle in a static spherically symmetric field are deduced; again these exhibit remarkable similarities to classical results. *H. Rund* (Durban).

Maurer-Tison, Françoise. L'espace fibré des corepères affines et son rôle fondamental en théorie unitaire d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **246** (1958), 240–243.

L'auteur définit dans une variété différentiable V_4 une connexion dite coaffine, par l'ensemble (g, ω) d'un champ de tenseurs deux fois covariants et d'une connexion linéaire, et une autre connexion $\bar{\omega}$ définie par

$$\bar{L}_{\rho\gamma}{}^{\beta}=L_{\rho\gamma}{}^{\beta}+g^{\alpha\beta}\nabla_{\gamma}g_{\alpha\rho}.$$

Ces deux connexions ont les mêmes géodésiques. Cette propriété permet l'usage en théorie d'Einstein-Schrödinger de la connexion coaffine qui unifie le tenseur fondamental et la connexion linéaire. *J. Renaudie* (Rennes).

See also: Quantum Mechanics: Salecker and Wigner; McLennan; Nakano; Omnes; Sokolov.

Astronomy

★**Kurth, Rudolf. Introduction to the mechanics of stellar systems.** Pergamon Press, New York-London-Paris, 1957. ix+174 pp. $9.00.

This outstanding little volume from the pen of Rudolf Kurth represents a valuable contribution indeed to the study of its subject, and a distinct addition to the contemporary astronomical literature. Its contents, divided in six chapters, embrace a rather wide choice of topics, ranging from a descriptive account of natural stellar systems such as galaxies or star clusters (chapter I) and an introductory survey of their mechanics (chapter II), to a mathematically rigorous and exacting account of the mathematical principles involved in the mechanics of mass-point systems (chapter III), and their statistical mechanics (chapter VI). Chapters IV and V, devoted to a discussion of stellar systems considered as assemblies of mass particles or self-graviating continua, constitute probably the most valuable part of the book — not perhaps because of any concrete new results, but rather for the penetrating light which they focus on basic principles of the methods involved. Of particular value, in this connection, is the author's account of Hopf's theorems (sec. IV-2) on the dynamics of mass-point systems, whose bearing on the subject has received so far but scanty attention on the part of most astronomers. Most chapters are concluded with selections of carefully formulated problems for the reader, which distinctly enhance the value of the book, as do the rather extensive bibliographical comments.

The only criticism which the present writer wishes to raise concerns the title of this slender volume — or, in particular, the use of the word "introduction", for the foregoing account of its contents, alone, indicates that the title cannot be interpreted very literally. At its least, the author's preoccupation with fundamental concepts and insistence on mathematical rigor of their presentation conceals, rather, the fact that the actual stellar systems, as we know them in the sky, are not yet fully amenable to treatment by his methods — which, in most cases, permit only very general conclusions to be drawn, without much specific content of interest for the practical astronomer. By its originality and insistence on rigorous formulation of basic concepts, Kurth's book represents without doubt a valuable contribution to the existing theoretical literature, and should be read as such by the mathematicians and physicists alike. It constitutes, however, scarcely an introduction; and the reviewer doubts if it can be studied with profit unless the reader has already a fair previous acquaintance with its subject on a more elementary plane, and is equipped with a mathematical knowledge rather far in excess of that postulated by the author in his introduction. Probably few readers would agree that the acquaintance with the concept of Lebesgue measure necessary to follow Kurth's account of Poincaré's recurrence theorem in section III–3, or of the theory of point sets employed in section IV–2 in connection with Hopf's theorems, lies "not much beyond the elements of the differential and integral calculus and of analytical geometry", as the author hopefully puts it in the foreword. Let us hope that this will become true in the future, but it is certainly not in our own days! *Z. Kopal.*

Brumberg, V. A. Permanent configurations in the problem of four bodies and their stability. Astr. Ž. **34** (1957), 55–74. (Russian. English summary)

In the paper under review the stability of the permanent configurations in the problem of four bodies is investigated for certain particular cases. In Section 1 is given an analysis of the Lagrangian solutions in the four body problem, based on the works of H. Andoyer [Bull. Astr. **23** (1906), 50–59], G. Meyer [Ann. Observ. Bordeaux **17** (1933), 77–252], W. D. MacMillan and W. Bartky [Trans. Amer. Math. Soc. **34** (1932), 838–875]. M. F. Subbotin [A course of celestial mechanics, vol. 2, Gostehizdat, Moscow-Leningrad, 1937, ch. 6] F. R. Moulton et al. [Periodic orbits, Carnegie Inst. Washington, 1920, ch. 8]. In Section 2 the results obtained by Andoyer [Bull. Astr. **23** (1906), 129–146] and Meyer concerning the stability of the circular orbits in the problem of n bodies are applied to the four-body problem. Section 3 deals with some particular circular solutions. Finally, in Section 4, an unsuccessful attempt is made to apply the previously developed theory of permanent configurations to the trapeza made up by four stars of the Orion configuration. *E. Leimanis* (Vancouver, B.C.).

Kurth, R. On Lagrange's triangular solution of the problem of three bodies. Arch. Math. **8** (1957), 381–392.

In the case of the Lagrangian triangular solutions the three particles form an equilateral triangle at each instant and move in the same fixed plane as if each of them were attracted only by the center of mass of the system, this attraction being in accordance with Newton's law of gravitation. The author questions whether or not Lagrange's triangular solutions are restricted to gravitating systems. It is shown in the paper that this kind of solution exists for all kinds of central forces. There are certain generalizations of the Lagrangian solutions to systems of more than three bodies. The author shows that they remain valid for all forces proportional to any power r^n of the distance r of two particles acting upon each other, except the case $n=-3$. Whether these generalizations also hold for other kinds of central forces remains unanswered. *E. Leimanis* (Vancouver, B.C.).

Torgård, Ingrid. Studies on particle orbits in plane models of stellar systems. Medd. Lunds Astr. Obs. Ser. II. no. 133 (1956), 91 pp.

The aim of the paper under review is to attempt a systematic study of the orbits of mass-particles which move in the plane of our galactic system (or of other galaxies), in the field of force governed by its gravitational potential. The ultimate aim of such work is to infer the specific form of this potential (i.e., the concentration of mass in the galactic plane with diminishing distance from the center) from the observed characteristics of motions (mainly in the radial direction) of different classes of distant galactic objects. In order to approach a solution of this inverse problem, the author sets out to construct numerically a considerable number of plane trajectories of mass-particles moving in a field of force of assumed (plausible) characteristics, integrated with the aid of the Stockholm electronic computer (BESK). These are exhibited graphically, and used as a basis for extended discussion of their various characteristics. *Z. Kopal.*

★ **Chandrasekhar, S. An introduction to the study of stellar structure.** Dover Publications, Inc., New York, N. Y. 1957. ii+509 pp. $2.75.

This book, whose original edition of 1939 was long out of print, has now been republished in economy form; and this event affords a suitable opportunity for assessing the significance of its contents from the way in which it has withstood the test of the time. The whole book has been divided into 12 chapters dealing with various aspects of the theory of self-gravitating gas spheres in radiative or convective equilibrium (non-equilibrium configurations being left out completely), but the promise of its title to provide an introduction to the study of stellar structure should not be taken too seriously; for whenever the author's interest gets aroused, the text is likely to become rather heavy going.

Judged from the vantage point of a contemporary reader, the first two chapters dealing with the laws of thermodynamics and formal aspects of the physical principles involved retain, on the whole, their interest, and so do chapters III on integral theorems (since little work has been done in this field since 1939) and V (the theory of radiation). Chapter IV (polytropic and isothermal gas spheres) contains by far the most comprehensive treatment of its subject available in English literature, and the actuality of the whole book leans heavily on this fact, for the rest of the book (chapters VI-XII, dealing with the models of actual stars) is more or less badly out of date (and so are the numerical data of the appendices). This is particularly true of the concluding chapter XII (stellar energy), which reflects virtually the pre-history of its subject matter, and which was out of date even when first published. Its comparison with the present state of the subject reveals more forcefully than anything else the tremendous advances made in this field in the past 20 years.

On the whole, the present value of the Dover reprint of Chandrasekhar's book of 1939 without change rests predominantly on its chapter IV, to which but little of significance has been added in the past two decades, and which still remains the most informative source of our knowledge of the formal aspects of polytropic gas spheres for those readers to whom Emden's classical book Die Gaskugeln remains inaccessible because of the language barrier. It is also a matter of some gratification to note that Chandrasekhar's book, whose original edition was priced at $ 10.00 by the University of Chicago Press 20 years ago, can now be made available in an entirely sufficient edition at $ 2.75. *Z. Kopal* (Manchester).

Ramakrishnan, Alladi; and Vasudevan, R. On the distribution of visible stars. Astrophys. J. 126 (1957), 573–578.

The number of stars per unit solid angle which are brighter than a specified apparent magnitude m_0 is given by

$$N(m)=\int_0^\infty D(\tau)\Phi(M)\tau^2 d\tau, \tag{1}$$

where τ is the distance from the observer along the line of sight measured in parsecs; $D(\tau)$ is the density of stars at τ; $\Phi(M)$ is the distribution function governing the occurrence of stars brighter than m which is related to M by

$$M=m+5-5\log\tau-a(\tau); \tag{2}$$

and $a(\tau)$ is the absorption in magnitudes by the intervening interstellar matter given by

$$a(\tau)=\sum_{i=1}^{n(\tau)}\varepsilon_i. \tag{3}$$

In (3), ε_i is a random variable representing the absorption by the ith cloud in the line of sight and $n(\tau)$ is the number of such interstellar clouds to the distance τ. All the ε_i's have the same distribution function; it is determined by the frequency of occurrence $\psi(q)$ of clouds with a transparency factor q. Further, $n(\tau)$ is governed by a Poisson distribution with a variance proportional to τ. In the earlier treatment of this problem by Chandrasekhar and Münch [Astrophys. J. 113 (1951), 150–165; MR 12, 644], the distribution of stars was assumed to be deterministic. In the present paper, an additional random element is introduced by assuming that $D(\tau)$ represents only the mean density and that a full description of the space distribution of the stars requires the complete sequence of "product densities" [Alladi Ramakrishnan, Proc. Cambridge Philos. Soc. 46 (1950), 595–602; MR 14, 296] $f_n(\tau_1, \tau_2, \cdots, \tau_n)$ which govern the probability of occurrence of a star at $(\tau_1, \tau_1+d\tau_1)$, another at $(\tau_2, \tau_2+d\tau_2)$ etc., irrespective of the number of stars elsewhere. With these assumptions, expressions are obtained for the various moments of the stochastic variable $N(m)$; and in the special case when $f_n(\tau_1, \cdots, \tau_n)=f_1(\tau_1)\cdots f_1(\tau_n)$ and $f_1=D_0\varepsilon^{-\tau/h}\tau^2$, explicit formulae are derived for the first two moments in terms of the incomplete gamma function. *S. Chandrasekhar* (Williams Bay, Wis.).

Nahon, Fernand. Sur la détermination du nuage des vitesses spatiales à partir des vitesses radiales; étude des grandes vitesses. C. R. Acad. Sci. Paris 244 (1957), 2688–2691.

Consider a distribution of stars in a plane. Given the distribution function of stars with different radial velocities ($R=x\cos\alpha+y\sin\alpha$, where x and y are velocities in two directions at right angles and α is an azimuthal angle which is the longitude) and longitudes, what is the distribution function governing the spatial velocity $V=(x^2+y^2)^{\frac{1}{2}}$? The solution of this problem originally obtained by Ambarzumian [Monthly Not. Roy. Astr. Soc. 96 (1936), 172–179] is rederived. The essential point in the solution is the observation that the function

$$\Phi(R, T)=\int_0^{2\pi}\frac{1}{2\pi}f(R\cos\alpha-T\sin\alpha, R\sin\alpha+T\cos\alpha)d\alpha$$

(where $f(x, y)$ is the distribution function of stars with

velocities x and y) is dependent only on R^2+T^2 if the spatial distribution is independent of α.

S. Chandrasekhar (Williams Bay, Wis.).

Lautman, Don A. On the atmospheric pulsations of cepheids. Astrophys. J. **126** (1957), 537–549.

The author considers the atmospheric pulsations of cepheids by means of a numerical calculation. He solves the exact equations of hydrodynamics for large amplitude oscillations. He does assume idealized thermo conditions. The atmosphere is assumed initially to be in equilibrium and an oscillation is started at the bottom. The calculation proceeds until the motion of the whole is periodic.

H. H. Goldstine (Princeton, N.J.).

Frank-Kamenetskii, D. A. On the spatial amplitude distribution in a pulsating star. Voprosy Kosmog. **5** (1957), 123–148. (Russian. English summary)

The non-adiabatic oscillation equations are investigated for "quasi-adiabatic" solutions, i.e., entropy wave amplitude small compared to displacement amplitude. The entropy wave amplitudes are found to grow without bound as the center is approached. The situation is more involved for the displacement amplitudes. It appears that there are circumstances in which these amplitudes decrease as the center is approached.

R. G. Langebartel (Urbana, Ill.).

Zhevakin, S. A. Central and peripheral theories of cepheid oscillations. Voprosy Kosmog. **5** (1957), 84–122. (Russian. English summary)

The author sets up boundary conditions at the periphery and center of a star in order that the solutions to the differential equations for non-adiabatic oscillations of a polytropic model be regular throughout the star. He criticizes the theory of Frank-Kamenetskii for using boundary conditions only at the periphery, thus giving rise to solutions that are not bounded near the center.

R. G. Langebartel (Urbana, Ill.).

Sen, K. K. Method of trigonometrical series in calculating the modification of intensity of monochromatic radiation due to multiple Compton scattering in stellar atmosphere. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 50–57.

The radiation at the photosphere level is assumed representable by the δ-function. The expansion through the second power in the Compton wave length of the scattering function for the atmosphere is used in the radiation transfer equation, which is then handled by a Fourier type analysis. A numerical example shows that the second order theory gives results sizably different from those of the first order theory. *R. G. Langebartel.*

Auluck, F. C.; and Kothari, D. S. On the change in shape of a gravitating sphere subject to the influence of a magnetic field. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 100–107.

A non-rotating self-gravitating sphere consisting of an infinitely conducting perfect fluid is set in a magnetic field which is uniform inside the sphere, uniform outside the sphere in the neighborhood of the sphere, and vanishing at infinity. A Legendre function type deformation is applied to the boundary, and the ellipticity for the resulting equilibrium spheroid is obtained. The results encompass those of Chandrasekhar and Fermi [Astrophys. J. **118** (1953), 116–141; MR **15**, 168] and Gjellestad [ibid. **120** (1954), 172–177; MR **16**, 183]. *R. G. Langebartel.*

Gjellestad, Guro. On the equilibrium of an oblate liquid spheroid with a magnetic field. Astrophys. J. **126** (1957), 565–572.

It was shown by Ferraro [Astrophys. J. **119** (1954), 407–412; MR **16**, 183] that an axisymmetric poloidal magnetic field (described in terms of a stream function U) which prevails in an equilibrium configuration of uniform density must satisfy the equation

$$\nabla^2 U=\varpi^2 f(U)$$

where ϖ denotes the distance from the axis and $f(U)$ is an arbitrary function of U; and outside the configuration $\nabla^2 U=0$, corresponding to a vacuum field. The solution of these equations in spheroidal coordinates for the case when $f(U)=\kappa=$a constant, is obtained. The solution involves two Legendre functions (in contrast to the solution in spherical polar coordinates which involves only one Legendre function). However, when the requirement that on the free surface of the configuration $\rho\Omega+\kappa U/4\pi$ (where Ω denotes the gravitational potential) is a constant is imposed, it is found that it cannot be met. For configurations of small eccentricities and weak magnetic fields, Ferraro's earlier results are recovered.

S. Chandrasekhar (Williams Bay, Wis.).

Javorskaya, I. M. The oscillations of an infinite self-gravitating gas cylinder in the presence of a magnetic field. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 988–990. (Russian)

The Eulerian equations governing radial motions of a cylindrical distribution of gaseous matter with a prevalent magnetic field, H, along the axis of the cylinder are

$$\frac{\partial^2 r}{\partial t^2}=-\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{2Gm}{r}-\frac{1}{8\pi\rho}\frac{\partial H^2}{\partial r},$$

$$\rho=\rho_0(r_0)\frac{r_0}{r}\frac{\partial r_0}{\partial r},\ p=p_0(r_0)\frac{\rho^\gamma}{\rho_0^\gamma},\ H=H_0(r_0)\frac{r_0}{r}\frac{\partial r_0}{\partial r},$$

$$m=m_0=2\pi\int_0^{r_0}\rho_0(r_0)r_0\,dr_0,$$

where ρ is the density, p is the pressure, m is the mass interior to r_0 and G is the constant of gravitation. It is shown that for the case $\rho_0=$constant; $p=\frac{1}{2}A\rho_0 r_0^2+$ constant, $H_0^2=4\pi B\rho_0 r_0^2+$constant, where

$$A=\frac{1}{\rho_0 r_0}\frac{\partial p_0}{\partial r_0}\text{ and }B=\frac{1}{8\pi\rho_0 r_0}\frac{\partial H_0^2}{\partial r_0},$$

the equation of motion allows a solution of the form

$$r=r_0\mu(t),\ v=r_0\mu'(t),\ \rho=\rho_0\mu^{-2}(t),$$
$$p=p_0\mu^{-2\gamma}(t),\ H^2=H_0^2\mu^{-4}(t),$$

where

$$t=\pm\int d\mu/\sqrt{f(\mu)},$$

$$f(\mu)=\frac{A}{\gamma-1}\mu^{-2(\gamma-1)}+B\mu^{-2}-\frac{4Gm_0}{r_0^2}\log\mu+\text{constant}.$$

Various physically distinct situations which arise from considering special cases of this solution are discussed.

S. Chandrasekhar (Williams Bay, Wis.).

Lidov, M. L. Automodel motions of a gas with spherical symmetry in a field with a gravitating center. Astr. Ž. **34** (1957), 603–608. (Russian. English summary)

Using the intuitive dimensional approach developed by L. I. Sedov [Similarity and dimensional methods in mechanics, 3rd ed., Gostehizdat, Moscow, 1954; MR **17**, 909], the author considers all possible "automodel" (i.e.

self-similar) nonstationary flows in a spherically symmetrical gas with a gravitating centre (Roche Model). The Roche Model may be characterised in the equilibrium state by two parameters A and μ, of independent dimensions: $\mu=GM$, therefore $[\mu]=[L]^3[T]^{-2}$; $\rho_0=A\mu^{s/2}/r^{3/2s+k+3}$, therefore $[A]=[M][L]^k[T]^s$. Here M is the mass of the core, and s, k are arbitrary constants. Since the independent variables of the problem are the time t and the distance r to the centre, the similarity variable is given by the dimensionless combination $\lambda=\mu^{-\frac{1}{2}}rt^{-\frac{1}{2}}$. Similarly, dimensional considerations allow the velocity, density and pressure of the flow to be expressed as products of dimensionless functions of λ with powers of A, μ and t. Substitution of these expressions in the hydrodynamical equations permits a reduction to ordinary differential equations. The flow is assumed to be bounded by a shock wave whose law of motion must evidently be $r^*=\mu^{\frac{1}{2}}t^{\frac{1}{2}}\lambda^*$, λ^* being an arbitrary constant.

The author then adopts the usual (and somewhat artificial) procedure of determining the manner in which the flow must be sustained for it to remain self-similar. Thus, the inner boundary condition might be given by supposing the core to move like a piston, according to the law $r=\alpha\mu^{\frac{1}{2}}t^{\frac{1}{2}}$, or else that the motion is due to an energy release proportional to a power of the time,

$$E=\beta A\mu^{(2-k)/3}t^{-(s+3k+3)},$$

where α and β are dimensionless constants.

One integral of the equations of motion is obtained from the condition that the motion remains adiabatic, while another integral is shown to exist when the energy is independent of the time. In this case, the similarity condition requires the special model $\rho_0=A\mu^{s/2}/r^2$, which was investigated by Carrus, Fox, Haas and Kopal [Astrophys. J. 113 (1951), 193–209; MR 12, 643]. The author obtains an analytical solution of this problem for $\gamma=4/3$, and examines the way in which the properties of the flow vary with the strength of the explosion (given by β). The density at the centre may be zero, finite or infinite, and the constant λ^* (which determines the shock speed) is found, as might be expected, to increase with β.

J. Hazlehurst (Manchester).

See also: Numerical Methods: Sheldon, Zondek and Friedman. **Mechanics of Particles and Systems:** Ogorodnikov. **Optics, Electromagnetic Theory, Circuits:** Giovanelli.

Geophysics

Jeffreys, Harold; and Vicente, R. O. The theory of nutation and the variation of latitude. Monthly Not. Roy. Astr. Soc. **117** (1957), 142–161.

The aim of the present paper has been to re-open the theory of the bodily tide and of various nutations of the terrestrial globe, in which elasticity of the shell and fluidity of the core are simultaneously taken into account. The model used for the shell is Takeuchi's Model 2 [Trans. Amer. Geophys. Union **31** (1950), 651–689], in which the core is replaced by a homogeneous incompressible fluid, with an additional particle at the center chosen so as to make the mass and moment of inertia of the core correct. Effects of the oceans are neglected.

The period of the free (Eulerian) nutation is found to be 392 days (and would be increased by the action of the ocean). The bodily tide numbers for semi-diurnal and long-periodic tides are: $h=0.58$, $k=0.29$, and $l=0.082$. The correcting factor for the 19-yearly nutation is 0.9964.

Z. Kopal (Manchester).

Jeffreys, Harold; and Vicente, R. O. The theory of nutation and the variation of latitude: the Roche model core. Monthly Not. Roy. Astr. Soc. **117** (1957), 162–173.

A theory developed by the authors in the paper reviewed above is modified by the use of terrestrial core whose density is a quadratic function of the radius (the variation of density being wholly due to pressure effects). Three pairs of cubic terms are introduced into the displacements. The period of the Eulerian nutation of the Earth built on this model would be 395 days, and the amplitude of the 19-year nutation would be multiplied by a factor of 0.9975. Allowance for the effect of the oceans would increase the former period to about 430 days, i.e., to a tolerable agreement with the observed facts. Correction factors are also given for the other principal terms of nutation. Those for the fortnightly nutation do not differ much from those appropriate for a central-particle model. There are substantial changes for the semi-diurnal nutations, owing to changes in free periods that occur in their neighbourhoods. The 19-year nutation in longitude remains, however, anomalous. *Z. Kopal* (Manchester).

Lauwerier, H. A. The influence of a homogeneous wind upon an infinitely wide North Sea. Math. Centrum Amsterdam. Afd. Toegepaste Wisk. Rep. **TW 41** (1957), 31 pp.

The author's objective is to study the effect of an arbitrary windfield upon the surface elevation of the North Sea. The analysis is based upon the linearized shallow water theory with friction and Coriolis effects considered. The model for the North Sea is taken to be an ocean of finite length b, and depth h, but of infinite width (the strip $\infty<x<\infty$, $0<y<b$). The wind is assumed to be uniform, unidirectional (perpendicular to the coast $y=0$), and time dependent only. The basic equations reduce to

$$\left(\frac{\partial}{\partial t}+\lambda\right)u-\Omega v=0,$$

$$\left(\frac{\partial}{\partial t}+\lambda\right)v+\Omega u+g\frac{\partial\xi}{\partial y}=\frac{V}{\rho h},$$

$$\frac{\partial}{\partial y}(hv)+\frac{\partial\xi}{\partial t}=0,$$

where u, v are the x and y velocity components, λ is a friction coefficient, g is the gravitational constant, Ω is the Coriolis coefficient, ξ is the wave height, and V is the component of tangential stress on the ocean surface due to the wind. The boundary conditions are $v=0$ for $y=0$ and $=0$ for $y=b$. The system is solved by Laplace transform methods and the natural modes, as well as the surface elevations due to periodic and step function winds, are obtained and discussed. It is noted that effect of the small damping factor is considerable. *H. Greenspan.*

Lauwerier, H. A. Exponential windfields. Math. Centrum Amsterdam. Afd. Toegepaste Wisk. Rep. **TW 42** (1957), 9 pp. (2 plates)

This report is a continuation of the above analysis. Experimental windfields which more closely simulate real storms are considered. *H. Greenspan.*

Fortak, H. Staubtransporte über staubaktiver Erdoberfläche. Z. Meteorol. **11** (1957), 19–27.

Es werden atmosphärische Staubtransporte in Abhängigkeit von der Beschaffenheit der Unterlage und unter Berücksichtigung des Schwereausfalls der Staubpartikel theoretisch untersucht.

Zusammenfassung des Autors.

Saltzman, Barry. Equations governing the energetics of the larger scales of atmospheric turbulence in the domain of wave number. J. Meteorol. **14** (1957), 513–523.

The author states the Fourier transforms of the equations of motion of the atmosphere, and uses them in a discussion of the energy balance of the system.

M. H. Rogers (Shrivenham).

Sigtryggsson, Hlynur; and Wiin-Nielsen, Aksel. Experiments in numerical forecasting, using space-smoothed fields. Tellus **9** (1957), 296–312.

Fogagnolo Massaglia, Bruna. Sulla rappresentazione delle onde sismiche della fase di Rayleigh. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956–57), 20–39.

In questa nota vengono determinate delle soluzioni delle equazioni dell'elasticità, che rappresentano la propagazione di onde elastiche in uno strato omogeneo, sovrapposto ad un mezzo omogeneo indefinito. Tali soluzioni costituiscono la generalizzazione delle ben note soluzioni di Love e permettono di seguire la propagazione di onde sismiche anche in prossimità dell'epicentro.

Author's summary.

Rikitake, Tsuneji. Oscillations of a system of disk dynamos. Proc. Cambridge Philos. Soc. **54** (1958), 89–105.

Bullard has previously studied the oscillations of a disk dynamo under varying conditions of drive, load, and parameters of the dynamo. In this work, two such dynamos are coupled together, and the oscillations are studied. It is found that the system is capable of reversals of current and magnetic field, such as may have occurred in the case of the earth's magnetic field. *H. W. Lewis.*

Camps, F.; et Lepropre, Melle M. Influence de la distorsion d'un objectif en aérotriangulation et en aéronivellement. Bull. Soc. Roy. Sci. Liège **26** (1957), 299–313.

In an aerial triangulation, which is constructed by extrapolating camera position and camera orientation forward through a chain of overlapping photographs, there is a tendency arising from lens distortion for the calculated orientation of the camera to pitch forward or backward progressively with respect to the true orientation. This pitching entails errors in the calculated position in the X direction (along the axis of the chain) and the vertical (Z). These well-known results [E. Loud, Manual of photogrammetry, 2nd ed., Amer. Soc. Photogr. Washington, D.C., 1952, p. 707] are shown theoretically to hold for the graphical method of aerial triangulation of Pauwen, which relies on numerous points in each stereoscopic pair of photographs; and the predicted errors are verified quantitatively by experiment. *J. A. O'Keefe.*

Bragard, L. La masse de la terre et sa densité moyenne. Bull. Soc. Roy. Sci. Liège **26** (1957), 290–294.

The product km' of the mass of the earth by the universal constant of gravitation is determined by integrating the acceleration of gravity over the whole sealevel surface of the earth (geoid), whose form is expressed in surface spherical harmonics. The resulting expression is

$$km' = a(K_0' - K) + a^2G\left(1 + \frac{3}{2}m - \varepsilon - \frac{15}{14}m\varepsilon + y_0\right),$$

where $K_0' - K$ is the difference of geopotential between a theoretical ellipsoid of revolution and the geoid, G is the equatorial value of gravity, ε the earth's oblateness, y_0 the zero-order surface spherical harmonic expressing the height of the geoid above the standard ellipsoid, and m a well-determined small constant (approximately the ratio of centrifugal force to gravity at the equator).

This direct determination of the constant of the terrestrial potential is compared with a determination by the methods of celestial mechanics. *J. A. O'Keefe.*

See also: Statistics: Linnik; Craddock; Charnock; Rushton and Neumann. **Elasticity, Plasticity:** Roseau. **Fluid Mechanics, Acoustics:** Schmidt.

OTHER APPLICATIONS

Economics, Management Science

★ **Papandreou, Andreas G. A test of a stochastic theory of choice.** University of California Publications in Economics, Vol. 16, No. 1. University of California Press, Berkeley-Los Angeles, 1957. 18 pp. $0.50.

One of the weaknesses of the theory of choice has been the failure to incorporate procedures to test theorems empirically. This the author (in collaboration with O. H. Sauerlender, O. H. Brownlee, L. Hurwicz and W. Franklin) sets out to remedy. From two axioms, (1) the existence of a relation which is transitive, reflexive, and connects the space of choices, (2) the existence of a commodity which can be used to compensate for the inferiority of any "bundle", the main theorem is derived. This states that if, for any three bundles A, B, C, A has a probability of being chosen over B at least equal to .5, and B has a probability of being chosen over C at least equal to .5, then the probability of choosing A over C is at least equal to the probability of choosing B over C. An elaborate test procedure was devised and applied (in two different forms) to 18 and 5 subjects respectively. The evidence of the test was strongly in favor of the theorem, although no strict rejection criterion is given.

G. Morton (Raleigh, N.C.).

van Klinken, J. Some remarks on dependent and independent probabilities. Verzekerings-Arch. Actuarieel Bijvoegsel **35** (1958), 13*–17*.

Der Verf. kritisiert die in der Versicherungs-Mathematik seit Karup bei einer Ausscheideordnung benutzte Terminologie der unabhängigen und abhängigen Wahrscheinlichkeiten und gibt an deren Welle stochastische Modelle im Sinne der modernen Wahrscheinlichkeitsrechnung. Der Verf. scheint die Ausführungen des Referenten in Versicherungsmathematik, T. I [Springer, Berlin, 1955; MR **17**, 168] über diesen Gegenstand nicht zu kennen.

W. Saxer (Zürich).

Giuliano, Salvatore. Premi in assicurazioni mediante derivate dei simboli di commutazione. Atti Accad. Peloritana Pericolanti. Cl. Sci. Fis. Mat. Nat. (3) **3(48)** (1945–49), 46–93 (1953).

Der Verf. betrachtet Versicherungsformen, bei denen die Versicherungsformen nach jedem Jahr steigen gemäss der Skala

$$1^r,\ 2^r,\ \cdots,\ n^r,\ r \text{ pos. ganze Zahl.}$$

Unter Annahme kontinuierlicher Verzinsung, definiert der Verfasser für $r=1, 2, 3, 4$ Kommutationszahlen und zeigt ihren Zusammenhang mit den Funktionen aleph von Wronski und den Euler'schen Differenzen $\Delta^s 0^r$. Numerische Tabellen. *W. Saxer* (Zürich).

★**A comprehensive bibliography on operations research through 1956, with supplement for 1957.** Publications in Operations Research, No. 4. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1958. xi+188 pp. $6.50.

The field of Operations Research impinges on so many other areas of science and finds outlet in so many different publications that, until a regular abstracting service is instituted, it will be most difficult for workers in the field to be sure they are up to date in their knowledge of contemporary progress. Consequently, general bibliographies are of considerable value and a number of them have been published. This one is the most up to date and complete of those used by the reviewer. The coverage is through 1957, the listing is by author's name, but a suggested code by subject is appended to each title so that, if each entry were punched on a card, detailed sorting by subject would be easy. In addition, this sorting has been done for about 40 subjects of general interest, the results comprising a set of "special bibliographies", which should be useful. As with all such bibliographies, the border-line between what should be excluded or included is difficult. The reviewer found no gaps in the entries which he felt were serious omissions, however.

P. M. Morse (Cambridge, Mass.).

Pinkham, Roger. An approach to linear inventory-production rules. Operations Res. **6** (1958), 185–189.

A method is given, using the method of generating functions and Cauchy's integral, for finding the mean square deviation of successive terms in any sequence defined by a linear difference equation with constant coefficients. Using this method, optimal coefficients may be determined for an inventory rule which is linear in forecasts, stocks and production rates, when it is assumed that costs depend only on the mean square deviations of successive stocks and of successive production rates.

M. J. Beckmann (New Haven, Conn.).

★**Peck, L. G.; and Hazelwood, R. N. Finite queuing tables.** Publications in Operations Research, No. 2. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1958. xvi+3–210 pp. $8.50.

The application of queuing theory to problems of machine maintenance has become increasingly important during the past several years. The equations differ from the usual queuing equations in that the population is finite, so the rate of "arrival" depends on the number of machines out of order already. This volume contains a set of tables giving the probability D of a delay in servicing a machine and the mean fraction of machines not running (being serviced or waiting for service) as a function of N, the total population of machines; M, the number of service channels (number of repair crews); and of X, the ratio between mean servicing time T and the sum $T+U$ of servicing time plus mean free running time, for X from 0 to 0.950, for N from 4 to 250, for an appropriate set of values of M. *P. M. Morse* (Cambridge, Mass.).

O'Neill, R. R. Analysis and Monte Carlo simulation of cargo handling. Naval Res. Logist. Quart. **4** (1957), 223–236.

See also: Probability: Senju; White and Christie; Jogdeo and Chaudhuri; Mack, Murphy and Webb; Mack. **Programming, Resource Allocation, Games:** Baumol and Wolfe; Bierlein.

Programming, Resource Allocation, Games

★**Bellman, Richard. Dynamic programming and its application to variational problems in mathematical economics.** Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. VIII, pp. 115–138. McGraw-Hill Book Co., Inc., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1958. 153 pp. $7.50.

Dynamic programming, multistage allocation processes and smoothing processes are discussed for the discrete and the continuous cases. The latter involves variational methods. Problems of computation, including the use of digital computers, are indicated. Applications include: allocation processes, inventories, bottleneck processes. The discussion is informal and no proofs are given. [For additional results and proofs, see Bellman, Dynamic programming, Princeton, 1957, MR **19**, 820.]

G. Tintner (Ames, Iowa).

Baumol, William J.; and Wolfe, Philip. A warehouse-location problem. Operations Res. **6** (1958), 252–263.

This is a nonlinear programming problem. Let X_{ijk} be the quantity shipped from factory i via warehouse j to retail location k; C_{ijk} the cost of shipment including inventory cost; Q_i the quantity shipped from plant i; R_j the capacity of warehouse j; S_k the quantity required at destination k. The problem is to minimize total delivery cost $\sum_{i,j,k} C_{ijk}$ subject to the conditions: $\sum_{jk} X_{ijk}=Q_i$ (all goods must be shipped out of the factory); $\sum_{ik} A_{ijk}\leq R_j$ (A_{ijk} inventory, no warehouse capacity can be exceeded); $\sum_{ij} X_{ijk}=S_k$ (all cosumer demand must be met); $X_{ijk}\geq 0$ (shipments cannot be negative). The objective function is concave, if we assume:

$$\sum_{ijk} C_{ijk}=\sum_{ijk} K_{ijk}X_{ijk}+\sum_j W_j(\sum_{ik} X_{ijk})^q+\sum_j V_j r_j \quad (0<q<1),$$

where K_{ijk} is the cost of transportation and handling per unit, W_j the cost of storage per period, $r_j=1$ if $\sum_{ik} X_{ijk}>0$, zero otherwise, V_j the administrative cost per warehouse. An iterative procedure for the computation is indicated, which will lead to lower total cost but does not guarantee the location of the true minimum. Convexity and concavity of the objective function in nonlinear programming is discussed. *G. Tintner* (Ames, Iowa).

Miehle, William. Link-length minimization in networks. Operations Res. **6** (1958), 232–243.

Two mechanical devices, one using pulleys and strings and the other using a soap solution, are described which can be used to determine experimentally a solution to the famous problem of connecting a system of villages by a network of roads of minimum total length. An iterative procedure for solving the problem analytically is also indicated, but no proof of convergence is presented.

Bernard Epstein (Philadelphia, Pa.).

Crane, Roger R. Some recent developments in transportation research. Naval Res. Logist. Quart. **4** (1957), 173–181.

Bierlein, Dietrich. Spieltheoretische Modelle für Entscheidungssituationen des Versicherers. Bl. Deutsch. Ges. Versicherungsmath. **3** (1958), 461–469.

The author formulates two game-theoretical models for the solution of insurance problems. The first model is a zero-sum game between nature, who chooses a true mortality table, and the insurer, who, not knowing nature's choice, chooses a premium distribution. The optimal premiums are derived for several pay-off functions. The second model is a non-zero-sum game between two competing insurers. Solutions are given for the cooperative and non-cooperative game. *M. Dresher.*

Borges, Rudolf. Nicht-cooperative Spiele mit nichtlinearen Auszahlungsfunktionen. Bl. Deutsch. Ges. Versicherungsmath. **3** (1958), 471–465.

Using the Brouwer Fixed-Point Theorem, the author proves the existence of equilibrium points for non-cooperative n-person games with non-linear payoff functions. *M. Dresher* (Santa Monica, Calif.).

★ **Gillette, Dean. Stochastic games with zero stop probabilities.** Contributions to the theory of games, vol. 3, pp. 179–187. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

A stochastic game consists of a finite collection of positions among which two players pass according to jointly-controlled transition probabilities, with the payoff accumulating throughout the game. The author investigates the consequences of assuming that the game is of infinite length, i.e., the stop probability is zero at some or all positions.

Defining an average payoff function and a discounted payoff function for the infinite game, the author relates his game to the stochastic game with positive stop probabilities for which there exists a value using "stationary" strategies, i.e., mixed strategies for the infinite game arising from the same mixed strategy at each occurrence of a given member of the collection of finite games. It is shown that if all the finite games are games of perfect information, then the infinite game has a solution in stationary pure strategies. *M. Dresher.*

★ **Karlin, Samuel. On games described by bell shaped kernels.** Contributions to the theory of games, vol. 3, pp. 365–391. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

The author calls the kernel $K(x, y)$ of a game over the unit square bell-shaped if $K(x, y)=\phi(x-y)$, where $\phi(t)$ is a positive analytic regular Polya frequency function (the last four words mean that $\phi(t) \in L^1(-\infty, \infty)$, that for every choice of $x_1<\cdots<x_n$, $y_1<\cdots<y_n$, the determinant $\phi(x_i-y_j)$ is non-negative and, moreover, that this determinant is not identically zero in the $y_j[x_i]$ for any choice of $x_i[y_j]$). It is proved that in games with such a kernel, satisfying in addition $\phi'(0)=0$, the optimal strategies of both players are finite step functions and are unique. The proof is elementary but intricate. The dependence of the solutions of games with kernel $K(x, y)=\phi(\lambda(x-y))$ on the parameter $\lambda>0$ is also studied. Other games with similar types of kernels are also considered.

A. Dvoretzky (Jerusalem).

★ **Milnor, J.; and Shapley, L. S. On games of survival.** Contributions to the theory of games, vol. 3, pp. 15–45. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

A "game of survival" is an analogue of the classical "gambler's ruin" process, in which the transition probabilities are to some extent controlled by the players. Although a good deal of work has been done in the past on this important class of two-person games, this is the first paper to discuss a number of aspects of the problem in a rigorous fashion. The analysis, which is a bit complicated, depends upon an ingenious combination of game theory techniques and the concept of a semi-martingale, borrowed from the usual theory of stochastic processes.

R. Bellman (Santa Monica, Calif.).

★ **Berge, Claude. Topological games with perfect information.** Contributions to the theory of games, vol. 3, pp. 165–178. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

Previously announced results of the author [C. R. Acad. Sci. Paris **241** (1955), 455–457; MR **17**, 1223] on the existence of equilibrium points in a class of infinite games in a global extensive form are presented. For this class of games, topological properties of the $v_i(x)$, the best payoff that player (i) can expect in position x, are studied. This last investigation is motivated both by previous attempts to calculate $v_i(x)$ by partial differential equations or other functional equations and by the questions which have been raised as to whether the smoothness of $v_i(x)$ justified the formulation of the partial differential equations.

S. Sherman (Philadelphia, Pa.).

Nolfi, P. Spieltheoretische Betrachtungen zur Stummen Mora. Elem. Math. **12** (1957), 127–129.

See also: Elasticity, Plasticity: Dorn and Greenberg.

Biology and Sociology

Dunham, B. The formalization of scientific languages. I. The work of Woodger and Hull. IBM J. Res. Develop. **1** (1957), 341–348.

In this first paper of a series of three the author presents and discusses the formal systems of Woodger [The axiomatic method in biology, Cambridge, 1937] and Hull [Mathematico-deductive theory of rote learning, Yale Univ. Press, 1940]. He argues that these attempts have not led very far because they are "heavily dependent upon other areas of unformalized theory." He feels, however, "that the lack of achievement rests upon discernible grounds", and that it is therefore possible to

indicate "how future work can be undertaken more profitably." *S. Gorn* (Philadelphia, Pa.).

★**Le Lionnais, F. L'imitation de la pensée créatrice par les machines.** Université de Paris, 1957. 31 pp.

In this popular lecture the author discusses various mechanical simulations of adaptive and "intelligent" behaviour such as Grey Walter's "tortoise", the vocoder, Audrey, and mechanical translation of languages. He also discusses the mechanization of Markov processes in the production of poetry, music, commercial brand names, and detective stories. Finally, a discussion of the mechanization of the game "odd and even" leads him to the conclusion that machines can simulate the "esprit de finesse" but not "creative thought." He points out, however, that much less of human thought than one might imagine is creative, and that, if not enough of our thinking were banal, human communication would become impossible. *S. Gorn* (Philadelphia, Pa.).

Davison, B. The compatibility of the survival plateaux hypothesis with Gaussian distribution of population. Bull. Math. Biophys. **19** (1957), 241–246.

See also: Computing Machines: Friedberg. **Statistics:** Lord.

Information and Communication Theory

Linfoot, E. H. An informational measure of correlation. Information and Control **1** (1957), 85–89.

The author defines the 'informational coefficient of correlation" of random variables x and y as $r_1=(1-e^{-2r_0})^{\frac{1}{2}}$, where r_0 is [in effect] the Gel'fand -Yaglom amount of information about y given by x [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 3–52; MR **18**, 980]; only the case where (real) x and y have a joint density in E_2 is discussed. The author gives the following properties of r_1: (1) $0\leq r_1\leq 1$; (2) if x and y are independent, then $r_1=0$; (3) if x and y are functionally related, then $r_1=1$; (4) $r_1(x', y')=r_1(x, y)$for (unspecified) transformations of the form $x'=f(x)$, $y'=g(y)$; (5) if x and y are jointly normal with (ordinary) correlation coefficient ρ, then $r_1=|\rho|$.

S. P. Lloyd (Murray Hill, N.J.).

Shannon, Claude E. Certain results in coding theory for noisy channels. Information and Control **1** (1957), 6–25.

This paper is concerned with discrete, finite, memoryless channels with finite input and output alphabets [for basic definitions and terminology see Shannon, Bell System Tech. J. **27** (1948), 379–423, 623–656; MR **10**, 133]. Let M be the number of words, each word composed of n input letters. Then $Pr(v|u)$ is defined as the probability of receiving the word v if the transmitted word was u. If $P(u)$ represents the probability for the input word u and $Pr(u, v)$ the joint probability, then the mutual information (per letter) $I(u, v)$ is $n^{-1}[\log Pr(v|u)-\sum_u P(u)\,Pr(v|u)]$. Let $\rho(x)$ be the distribution function of the random variable I. The author shows that if P_e is the probability of error for a coding and decoding system, then for any M and any $\theta>0$ there exists a decoding system such that if all words have equal probability $1/M$, then $P_e\leq\rho(R+\theta)+e^{-n\theta}$, where $R=n^{-1}\log M$. An expression for the channel capacity of a finite state channel is also given, as well as certain generalizations and associated results on channel capacity and probability of error.

K. S. Miller (New York, N.Y.).

Miller, George A.; and Friedman, Elizabeth A. The reconstruction of mutilated English texts. Information and Control **1** (1957), 38–55.

This paper studies the effects of five types of mutilation on our ability to reconstruct the original message, and examines how much redundancy we can eliminate without modifying the basic rules of English orthography. The five types of mutilation were: (1) Substitution of characters chosen at random for some of the original characters, (2) Indicated substitution, where the place of the substitution is underlined, (3) Deletion, with the blank space left there, (4) Abbreviation, which is deletion plus closing up the space, and (5) Insertion of characters into the original message. Only one type was used at a time and used at levels of 10, 20, 30, 40 and 50%.

The results of their experiments are clearly presented and are analyzed both as to implications on how to cope with errors and as to the implied redundancy that can be handled. Typical of their observations is "it is better simply to delete and leave blank the incorrect character than to print it, or to abbreviate the message, or to print it with an indication that it is wrong."

They also note "The most successful procedure for coherent abbreviation that we have found is to omit the space and all the vowels:" which reminds one of some ancient written languages which did not write the vowels at all. *R. W. Hamming* (Murray Hill, N.J.).

★**Khinchin, A. I. Mathematical foundations of information theory.** Translated by R. A. Silverman and M. D. Friedman. Dover Publications, Inc., New York, N. Y., 1957. ii+120 pp. $1.35.

The articles in this book are: "The entropy concept in probability theory" [MR **15**, 238] and "On the fundamental theorems of information theory" [MR **17**, 1098].

Breiman, Leo. The individual ergodic theorem of information theory. Ann. Math. Statist. **28** (1957), 809–811.

It is proved that $\lim_n(-(1/n)\log_2 p(x_0, x_1, \cdots, x_{n-1})=H$ (the entropy) almost surely, for every stationary ergodic process $\cdots, x_{-1}, x_0, x_1, \cdots$ ranging over a finite alphabet $a_1, a_2, \cdots, a_s$. In this scope, the sequence $\{g_k(\omega)\}$ is defined by putting $g_0(\omega)=-\log_2 p(x_0)$, $g_k(\omega)=-\log_2(p(x_{-k}, x_{-k+1}, \cdots, x_0)/p(x_{-k}, x_{-k+1}, \cdots, x_1)$, $k\geq 1$, hence $-(1/n)\log_2 p(x_0, \cdots, x_{n-1})=(1/n)\sum_{k=0}^{n-1} g_k(T^k\omega)$, T being the shift operator, which is measure preserving and metrically transitive. The sequence $\{g_k(\omega)\}$ converges almost surely to a function $g(\omega)$, with $E(\sup_k g_k)<\infty$. The Birkhoff ergodic theorem applied to $g(\omega)$, yields $\lim(1/n)\sum_{k=0}^{n-1} g_k(T^k\omega)=\lim(1/n)\sum g(T^k\omega)=Eg$ almost surely.

O. Onicescu (Bucarest).

See also: Groups and Generalizations: Croisot. **Numerical Methods:** Booth. **Probability:** Slepian; Davenport and Root; Elldin; Blachman.

Control Systems

Ostrowsky, G. M. Application of nonlinear corrective devices in second order automatic control systems. Avtomat. i Telemeh. **17** (1956), 979–984. (Russian. English summary)

Consider a control system governed by the equation

$\ddot{e}+f(e)\dot{e}+e=0$, in which $f(e)$ is constrained by $c \leq f(e) \leq b$, b and c being given constants. The problem is to find an "optimal" $f(e)$, that is, an $f(e)$ which (i) results in no overshoot in response to step disturbance of any magnitude, and (ii) minimizes the transit time to equilibrium. By using elementary arguments, the author shows that $f(e)=c$ for $(\dot{e}+|\gamma_1|e)\dot{e}<0$ and $f(e)=b$ for $(\dot{e}+|\gamma_1|e)\dot{e}>0$, where $\gamma_1=-\frac{1}{2}b-\sqrt{(\frac{1}{4}b^2-1)}$. *L. A. Zadeh.*

Feldbaum, A. A. On application of computers to automatic control systems. Avtomat. i Telemeh. **17** (1956), 1046–1056. (Russian. English summary)

An expository account of the basic principles of optimal and adaptive control systems. *L. A. Zadeh.*

Batkov, A. M.; and Solodovnikov, V. V. A method for determining optimum characteristics of a certain class of adaptive control systems. Avtomat. i Telemeh. **18** (1957), 378–391. (Russian. English summary)

Consider a linear (not necessarily time-invariant) system characterized by its impulsive response $k(t, \tau)$. Assume (i) that the input is of the form $g(t)+n(t)$, where $g(t)$ is a given non-random signal and $n(t)$ is a stationary noise, and (ii) that the performance index is expressed by $E^2=\varepsilon_d^2(t)+\lambda^2(t)\varepsilon_r^2$, where $\varepsilon_d(t)$ is the non-random component of the error, ε_r^2 is the mean square value of the random component and $\lambda^2(t)$ is a specified weighting function. The authors derive an integral equation satisfied by a function $k(t, \tau)$ which minimizes E^2 in the class of all physically realizable impulsive responses, and obtain a general solution of this equation by using an extension of the Green's function technique of Dolph and Woodbury [Trans. Amer. Math. Soc. **72** (1952), 519–550; MR **14**, 295]. *L. A. Zadeh* (New York, N.Y.).

Andreev, N. I. Determination of an optimal linear dynamic system by using the criterion of extremum of a functional of particular form. Avtomat. i Telemeh. **18** (1957), 615–619. (Russian. English summary)

Consider a linear system characterized by its impulsive response $k(\tau)$. Assume that the criterion function is of the form $I=g(I_1, I_2, \cdots, I_{n+1})$, where g is a given real-valued function and the I_i $(i=1, 2, \cdots, n+1)$ are quadratic functionals of $k(\tau)$. The author shows that the determination of an impulsive response, $k_0(\tau)$, which extremizes I can be reduced to extremizing a quadratic functional of the form $I'=a_1I_1+\cdots+a_nI_n+I_{n+1}$, where the a_i are undetermined coefficients. Let $k_0'(\tau; a_1, \cdots, a_n)$ be an impulsive response which extremizes I'. Then the problem of extremizing I over the class of impulsive response functions reduces to that of finding the extremum of I' over the a_i $(i=1, \cdots, n)$. The exposition of the method is incomplete and lacking in rigor. *L. A. Zadeh.*

Savinov, G. V.; and Citovič, P. A. On a linear nonautonomous system. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 3, 9–12. (Russian)

The authors consider the differential equation $\ddot{x}+f(t)x=0$ and essentially obtain the W.K.B. solution of this equation. *L. A. Zadeh* (New York, N.Y.).

Meerov, M. V. On the structural synthesis of multi-loop control systems containing elements with pure time-lags. Avtomat. i Telemeh. **18** (1957), 1098–1108. (Russian. English summary)

This paper is an extension of earlier work by the same author [Trudy Sovešan. Teor. Avtomat. Reg., v. 1, Izdat. Akad. Nauk SSSR, Moscow, 1955] to the case of systems containing elements with pure time-lags. In the present paper, the author finds multi-loop structures which have the property that the system remains stable when its loop-gains increase indefinitely. It is shown that a system possesses this property if and only if every loop whose gain may increase indefinitely is structurally equivalent to a single-loop system which remains stable for indefinitely large values of the loop-gain. *L. A. Zadeh.*

Batkov, A. M. Concerning the problem of the synthesis of linear dynamic systems with variable parameters. Avtomat. i Telemeh. **19** (1958), 49–54. (Russian. English summary)

Let $W(t, \tau)$ be the impulsive response of a linear time-varying system characterized by the equation

$$L(p, t)u(t)=M(p, t)v(t),$$

where $v(t)$ is the input, $u(t)$ is the output, $p=d/dt$, and $L(p, t)$ and $M(p, t)$ are differential operators, $L(p, t)=\sum_{i=0}^{n} a_i(t)p^{n-i}$, $M(p, t)=\sum_{j=0}^{m} b_j(t)p^{m-j}$, in which the $a_i(t)$ and $b_j(t)$ are continuous functions of time. The author develops a method for determining $L(p, t)$ and $M(p, t)$ when $W(t, \tau)$ is given in the form $W(t, \tau)=\sum_{i=1}^{n} \varphi_i(t)\psi_i(\tau)$, and illustrates it by several simple examples. The solution of this problem provides a means for simulating the impulsive response of a lumped-constant system by the use of an analog computer. *L. A. Zadeh.*

Ludwig, G.; und Rollnik, H. Erzwungene Schwingungen und Fehler bei Regelsystemen mit zeitlich variabler Regelstärke. Z. Angew. Math. Mech. **38** (1958), 16–20. (English, French and Russian summaries)

This is a continuation of a previous paper by the same authors, dealing with the general theory of linear control systems which are so designed that the error is ideally reduced to zero at a prescribed instant [same Z. **37** (1957), 457–470; MR **19**, 1028]. Mathematically, the theory amounts to the theory of the differential equation $P(d/dt)(t\varepsilon)+Q(d/dt)\varepsilon=n(t)$, where $P(z)$ and $Q(z)$ are polynomials in z, $n(t)$ is a given function, and ε is the unknown. The previous paper dealt with the case in which $n(t)\equiv 0$; the present one deals with more general cases. In the auxiliary case in which $n(t)=e^{t\omega t}$, the authors obtain a particular solution of the form

$$\varepsilon(t)=\int_C G(z)e^{tz}dz,$$

where C is a suitable path of integration. The result is used in an obvious way to obtain a particular solution in the case in which $n(t)$ is given in the form

$$n(t)=\int_{-\infty}^{\infty} g(\omega)e^{t\omega t}d\omega.$$

The discussion does not go far beyond the obtaining of analytical expressions for these particular solutions. However, a few of the properties of the solutions, of importance for control theory, are discussed briefly.

L. A. MacColl (New York, N.Y.).

★ **Cosgriff, Robert Lien. Nonlinear control systems.** McGraw-Hill Series in Control Systems Engineering. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. viii+328 pp. $9.00.

This is an elementary textbook for the use of students in engineering schools. Successive groups of chapters deal with: (1) linear control systems; (2) various more or less reliable methods for studying nonlinear systems; (3) chiefly interested in the practical art of designing control (4) logic circuits as control elements. The author is random signals and applications of statistical methods; systems; and he treats this in elaborate detail, with a

wealth of illustrative material. On the other hand, the presentations of matters of fundamental theory tend to be sketchy and wanting in clarity and precision. Within its limitations, the book is competently written and it will probably meet a certain need adequately.

L. A. MacColl (New York, N.Y.).

★ **Zypkin, Ja. S. Theorie der Relaissysteme der automatischen Regelung.** R. Oldenbourg, Münich; Verlag Technik, Berlin; 1958. 472 pp. DM 52.00.

A translation by W. Hahn and R. Herschel from the Russian of the book reviewed in MR **18**, 709. The German edition is essentially unchanged.

Kavanagh, R. J. A control system interpretation of certain properties of matrices. Matrix Tensor Quart. **8** (1957), 32–38.

See also: Ordinary Differential Equations: Krasovskiĭ. **Computing Machines:** Shannon. **Probability:** Dostupov and Pougatchev.

HISTORY, BIOGRAPHY

Tosi, A. "De centro gravitatis solidorum" di Luca Valerio. Period. Mat. (4) **35** (1957), 189–201.

Nachbin, Leopoldo. Aspects of the recent development of mathematics in Brazil. Soc. Parana. Mat. Anuário **3** (1956), 28–41. (Portuguese)

Dilgan, H. Sur un problème indéterminé d'Ibni Hamza. Bull. Tech. Univ. Istanbul **10** (1957), no. 3, 1–5. (Turkish summary)

Levi, F. W. Gauss und das Raumproblem. Math.-Phys. Semesterber. **5** (1957), 191–199.

Haldane, J. B. S. Karl Pearson, 1857-1957. Biometrika **44** (1957), 303–313.

A centenary lecture.

Fladt, K. Zum hundertsten Geburtstage von Friedrich Schur. Math.-Phys. Semesterber. **5** (1957), 182–185.

Anonymous. Luigi Cremona. Archimede **9** (1957), 137–139. (1 plate)

Anonymous. Ulisse Dini. Archimede **9** (1957), 263–266. (1 plate)

Anonymous. Vito Volterra. Archimede **10** (1958), 29–32. (1 plate)

Viola, Tullio. Ugo Amaldi. Archimede **10** (1958), 33–37. (1 plate)

Anonymous. Obituary: Luigi Fantappiè. Collect. Math. **9** (1957), 3–5. (1 plate) (Spanish)

Verblunsky, S. Obituary: Robert Mark Gabriel. J. London Math. Soc. **33** (1958), 125–128.

Brun, Viggo. Carl Størmer in memoriam. Nordisk Mat. Tidskr. **5** (1957), 169–175, 213. (Norwegian. English summary)

Anonymous. Obituary: Heinrich Scholz. Math.-Phys. Semesterber. **5** (1957), 181.

Rider, P. R. Obituary: Otto Dunkel. Amer. Math. Monthly **64** (1957), no. 7, part II, 1–2.

This brief obituary of a long-time editor of the Problem Department of the Monthly accompanies a selection of the 400 "best" problems proposed in that Department between 1918 and 1950.

Behnke, Heinrich; und Hermes, Hans. Johann von Neumann, ein grosses Mathematikerleben unserer Zeit. Math.-Phys. Semesterber. **5** (1957), 186–190.

See also: Computing Machines: Shannon. **Quantum Mechanics:** Van Hove.

Mathematicians

FOR ANALYSIS GROUP of expanding Research & Development Laboratory. Principal fields of interest are weapons systems analysis, electromagnetic propagation, operations research, nuclear phenomena, probability and statistics.

Several openings are available for Mathematicians with masters or doctorates in mathematics or physics. These openings require men with vision and initiative.

Our modern laboratory provides a professional working atmosphere and the location in a quiet suburban area makes for pleasant living and working with easy access to the cultural and educational facilities of metropolitan New York and New Jersey. Liberal benefits include a tuition refund plan.

All inquiries in confidence. Please send resume including salary requirements to A. A. Franklin.

Vitro LABORATORIES
DIVISION OF
VITRO CORPORATION OF AMERICA
200 Pleasant Valley Way, West Orange, New Jersey

(Other laboratories located at Eglin AF Base, Florida, & Silver Spring, Maryland).

MEMOIRS 26

PRINCIPAL SOLUTIONS OF
ORDINARY DIFFERENTIAL EQUATIONS
IN THE COMPLEX DOMAIN

WALTER STRODT

The author investigates non-linear algebraic differential equations of the nth order, with coefficients asymptotically equivalent, in a sector of the complex plane, to logarithmic monomials, i.e. to products of real powers of x, $\log x$, $\log\log x, \ldots$ The author's *algorithm of the principal monomial* (cf. Memoirs 13) is applied in two ways–first: to transform the given equation into one which is quasi-linear; second: to produce an *approximate factorization* of that quasi-linear equation into first order linear factors.

$2.00 *170 pages, 1957*

25% discount to members of the Society

Order from
AMERICAN MATHEMATICAL SOCIETY
190 Hope Street, Providence, R.I.

THE LONDON MATHEMATICAL SOCIETY

The Society, which was founded in 1865, has for its objects **the promotion and extension of mathematical knowledge.** The current publications of the Society are the **Proceedings** (3rd series), which began in 1951, and the **Journal,** established in 1926. The Society maintains a high standard in its publications, and recent volumes contain memoirs and notes of great importance for the development of mathematics.

The **Proceedings** (3rd series) is published by the Oxford University Press, London E.C. 4, and the subscription is 105s. (about $15) annually. The **Journal,** which contains mainly shorter papers, is published by C. F. Hodgson and Son Ltd., Pakenham Street, London W.C. 1, and the subscription is 84s. (about $12) annually. Four parts of each are issued annually, in January, April, July and October. Subscription may be effected either with the publishers or through any bookseller.

Particulars of membership may be obtained from the Honorary Treasurer, London Mathematical Society, Burlington House, London W. 1, England. There is a reciprocity agreement with the American Mathematical Society, particulars of which will be sent on request. Both **Proceedings** and **Journal** are issued free to members.

Information concerning the submission of papers for publication, and concerning the purchase of back numbers, is given on the inside covers of current issues of both the **Journal** and **Proceedings,** and will also be supplied on request by Professor K. A. Hirsch (Honorary Secretary), Queen Mary College, London E. 1, England.

RUSSIAN TRANSLATIONS

SERIES II

Eleven papers on Topology, Function Theory, and Differential Equations

Volume 1.
By P. S. Aleksandrov, M. G. Kreĭn, G. E. Šilov, I. M. Gel'fand and S. V. Fomin, G. D. Suvorov, L. S. Pontryagin, I. I. Gihman, V. M. Staržinskiĭ, L. È. Reĭzin' and I. M. Gel'fand and B. M. Levitan. Published in 1955. 304 pages. $3.60

Eight papers on Group Theory

Volume 2.
By A. I. Malcev, L. Ya. Kuliov, O. N. Golovin, M. A. Naĭmark, I. M. Gel'fand and M. I. Graev, and I. M. Gel'fand and Z. Ya. Šapiro. Published in 1956. 316 pages. $3.70

Five papers on Logic, Algebra, and Number Theory

Volume 3.
By A. A. Zykov, D. K. Faddeev, V. I. Nečaev, Yu. V. Linnik and A. V. Malyšev, and A. Z. Val'fiž (Walfisz). Published in 1956. 246 pages. $3.30

Eight papers on Algebra and Number Theory

Volume 4.
By A. Z. Val'fiž (Walfisz), N. M. Korobov, and I. R. Šafarevič. Published in 1956. 237 pages. $3.80

Eight papers on Functional Analysis and Partial Differential Equations

Volume 5.
By S. Ya. Havinson, M. A. Naĭmark and S. V. Fomin, M. S. Livšic, I. M. Gel'fand and D. A. Raĭkov and G. E. Šilov, I. M. Gel'fand and G. E. Šilov, A. G. Kostyučenko and G. E. Šilov, V. M. Borok and È. S. Citlanadze. Published in 1957. 333 pages. $4.30

Five papers on Algebra and Group Theory

Volume 6.
By P. K. Raševskiĭ, E. B. Dynkin, M. A. Naĭmark, and N. Ya. Vilenkin. Published in 1957. 483 pages. $5.20

Two papers on Homotopy Theory of Continuous Mappings

Volume 7.
By M. M. Postnikov and V. G. Boltyanskiĭ. Published in 1957. 321 pages. $5.50

Twelve papers on Function Theory, Probability, and Differential Equations

Volume 8.
By K. Noshiro, A. P. Matorin, L. S. Pontryagin, I. M. Gel'fand and Z. Ya. Šapiro, A. M. Yaglom, I. M. Gel'fand and V. B. Lidskiĭ, A. F. Andreev, V. M. Volosov, V. K. Saul'ev, L. A. Lyusternik, and B. E. Bonštedt. Published in 1958. 356 pages. $6.00

Four papers on Group Theory

Volume 9.
By P. S. Novikov, I. M. Gel'fand and M. I. Graev, and M. A. Naĭmark. Published in 1958. 231 pages. $5.00

Thirteen papers on Analysis and Topological Algebraic Structures

Volume 10.
By A. F. Leont'ev, I. I. Pyateckiĭ-Šapiro, S. N. Mergelyan, O. S. Ivašev-Musatov, V. A. Yakubovič, I. G. Petrovskiĭ and E. M. Landis, M. I. Višik and O. A. Ladyženskaya, I. A. Gol'dfain, Yu. I. Lyubič, B. M. Gagaev, A. G. Sigalov, Yu. L. Šmul'yan, M. A. Krasnosel'skiĭ. Published in 1958. 409 pages. $6.60

25% discount to members of the American Mathematical Society

Order from

AMERICAN MATHEMATICAL SOCIETY

190 Hope Street, Providence 6, R.I

AN IMPORTANT ANNOUNCEMENT

Concerning

THE PROCEEDINGS OF THE SYMPOSIA IN APPLIED MATHEMATICS

These symposia were held under the auspices of the American Mathematical Society and other interested organizations. The Society itself published the first two volumes. The McGraw-Hill Book Company, Inc., published and sold Numbers 3 through 8.

By special arrangement with the McGraw-Hill Book Company, Inc., these six volumes have now been transferred to the American Mathematical Society. Orders henceforth should be placed through the Society.

Volume	Price
1. **Non-linear problems in mechanics of continua.** Editorial Committee: W. Prager, Eric Reissner, J. J. Stoker. 1949, viii + 219 pp. *Papers by*: Bergman, Bers, Birkhoff, Brinkley, Kirkwood, Carrier, Coburn, and Dolph, Drucker, Emmons, Friedrichs, Kaplan, Kuo, Lees, Levy, Lin, Murnaghan, Opatowski, Polachek and Seeger, Poritsky, Prager, Reissner, Sears, Stoker, Taub, Weinstein.	$5.25
2. **Electromagnetic theory.** Editorial Committee: R. V. Churchill, Eric Reissner, A. H. Taub. 1950, iv + 91 pp. *Papers by*: Duffin, Feenberg, Feshbach, Heins, Infeld, Kac, Lee, Pekeris, Ramberg, Rice, Stevenson, Synge, Taub, Truell, Wallman, Watson, Wiener.	$3.00
3. **Elasticity.** Editorial Committee: R. V. Churchill, Eric Reissner, A. H. Taub, 1950, vi + 233 pp. *Papers by*: Carrier and Shaw, Coburn, Friedrichs, Hay, Hildebrand, Holl, Lee, March, Perry, Poritsky, Prager and Symonds, Reissner, Seth, Sokolnikoff, Southwell, Stoker, Weinstein.	$6.00
4. **Fluid dynamics.** Edited by M. H. Martin. 1953, vi + 186 pp. *Papers by*: Birkhoff, Young and Zarantonello, Burgers, Busemann, Carrier and Yen, Chandrasekhar, Degroot, Heins, Lin, Martin and Thickstun, Meyer, Synge, Theodorsen, Thomas, Weinstein.	$7.00
5. **Wave motion and vibration theory.** Edited by A. E. Heins. 1954, vi + 169 pp. *Papers by*: Bleakney, Carrier and Munk, Chandrasekhar, Duffin and Schild, Garabedian, Heins and Feshbach, Lee, Lefschetz, Levine, Lin, Magnus, McLachlan, Montroll and Greenberg, Stoker, Weinstein.	$7.00
6. **Numerical analysis.** Edited by J. H. Curtiss. 1956, vi + 303 pp. *Papers by*: Bellman, Bergman, Bruck, Clutterham and Taub, Fischbach, Frankel, Hastings, Hayward and Wong, Hestenes, Lehmer, Motzkin, Rosenbloom, Sard, Taussky, Tompkins, Walsh, Warschawski, Wasow, Wielandt, Young.	$9.75
7. **Applied probability.** Edited by L. A. MacColl. 1957, v + 104 pp. *Papers by*: Batchelor, Dobb, Feller, Hoff, Kac, Koopman, Levy, Munch, Ulam.	$5.00
8. **Calculus of variations and its applications.** Edited by L. M. Graves. 1958, v + 153 pp. *Papers by*: Bellman, Chandrasekhar, Diaz, Drucker, Hodge, Keller, Reissner, Rothe, Shiffer, Synge, Weinberger.	$7.50
IN PREPARATION	
9. **Orbit theory.** Edited by R. E. Langer.	
10. **Combinatorial designs and analysis.** Edited by R. E. Bellman.	

25% discount to members of the Society

AMERICAN MATHEMATICAL SOCIETY

Order from

190 Hope Street, Providence 6, R.I.

Books of interest from
ADDISON-WESLEY

LIMIT DISTRIBUTIONS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

By B. V. Gnedenko and A. N. Kolmogorov. *Translated by K. L. Chung, Syracuse University, with additional appendix material provided by the translator and J. L. Doob, University of Illinois.*

From a review in *Zeitschrift Fur Angewandte Mathematik Und Physik*

"Every mathematician who works with modern limit distributions for calculating probabilities should become familiar with this book. It is a clear and axiomatic exposition of classical and modern limit distributions of probability calculation, along with the concepts pertinent to them, such as characteristic functions and infinitely divisible distributions".

264 pp., 1954, $ 12.00

THEORY OF MATRICES

By Sam Perlis, *Purdue University*

From a review in *The Matrix and Tensor Quarterly*

"It is easily discernible in this book that the underlying theme is fundamentals – the viewpoint which is presented is that of the pure mathematician and while the treatment is directed to the needs of students majoring in mathematics, those who are more interested in applications will have little difficulty in making practical use of the book. We recommend the book without any hesitation as a first-class study".

237 pp., 5 illus (1952) $ 6.50

AN INTRODUCTION TO MATHEMATICAL THOUGHT

By E. R. Stabler, *Hofstra College*
From a review in *Mathematical Reviews*

"One of the indirect values of an introductory treatment of this character is that it is likely to lead thoughtful students to questions, particularly as to the philosophical basis of mathematics, which might otherwise never occur to them".

268 pp., 5 illus (1953) $ 6.50

ANALYTIC AND PROJECTIVE GEOMETRY

By D. J. Struik, *Massachusetts Institute of Technology*

From a review in *Nieuw Archief Voor Wiskunde*

"The lively treatment is enhanced by a mixture method, by many terminological finds, by many original details, a great many examples from the geometry of the triangle and the tetrahedron, by the extensive historical and bibliographical notes and many exercises, accompanied partly by instructions and answers".

291 pp., 116 illus (1953) $ 7.50

DIFFERENTIAL GEOMETRY

By D. J. Struik, *Massachusetts Institute of Technology*

From a review in *Bulletin of The Mathematical Society*

"The comprehensive historical, biographical, and bibliographical information will convey to the student the enthusiasm of the pioneers in this field of research, put him in contact with the original ideas in their bare form (not overshadowed by symbolism, however powerful), and give him access to the more extensive treatises on his subject".

221 pp., 121 illus (1950) $ 7.50

ADDISON-WESLEY PUBLISHING COMPANY, INC.

Reading Massachusetts U.S.A.

THE ANNALS OF MATHEMATICAL STATISTICS

Vol. 30, No. 1 — March, 1959

Invariance Theory and a Modified Minimax Principle — Oscar Wesler

On Linear Associative Algebras Corresponding to Association Schemes of Partially Balanced Designs — R. C. Bose and Dale M. Mesner

On a Characterization of the Triangular Association Scheme — S. S. Shrikhande

On a Generalisation of the Kronecker Product Designs — B. V. Shah

Application of a Measure of Information to the Design and Comparison of Regression Experiments — M. Stone

Note on Estimating Information — Colin R. Blyth

Unbiased Sequential Estimation for Binomial Populations — Morris H. DeGroot

A Single-Sample Multiple-Decision Procedure for Selecting the Multinomial Event Which Has the Highest Probability — Robert E. Bechhofer, Salah Elmaghraby, and Norman Morse

A Property of the Multinomial Distribution — Harry Kesten and Norman Morse

A Classification Problem Involving Multinomials — Oscar Wesler

A Problem in Optimum Filtering with Finite Data — Gopinath Kallianpur

On the Limiting Distribution of the Number of Coincidences Concerning Telephone Traffic — L. Takács

On the Integrodifferential Equation of Takács. II — Edgar Reich

Symmetrizable Markov Matrices — H. P. Kramer

On Some Statistical Tests for Mth Order Markov Chains — Leo A. Goodman

Consensus of Subjective Probabilities: The Pari-Mutuel Method — Edmund Eisenberg and David Gale

A Note on Perfect Probability — Gopinath Kallianpur

On the Distribution of the Kolmogorov-Smirnov D-Statistic — Pedro Egydio de Oliveira Carvalho

Equality of More Than Two Variances and of More Than Two Dispersion Matrices Against Certain Alternatives — R. Gnanadesikan

On the Theory of BAN Estimates — Robert A. Wijsman

Estimation of the Medians for Dependent Variables — Olive Jean Dunn

A Consistent Estimator of a Component of a Convolution — William R. Gaffey

Bayes and Minimax Procedures in Sampling from Finite and Infinite Populations — Om P. Aggarwal

An Approximation Useful in Univariate Stratification — Gunnar Ekman

Truncation and Tests of Hypotheses — Om P. Aggarwal and Irwin Guttman

Notes:

Bartlett Decomposition and Wishart Distribution — A. M. Kshirsagar

Some Bounds on the Distribution Functions of the Largest and Smallest Roots of Normal Determinental Equations — Ray Mickey

Note on a Moving Single Server Problem — S. Karlin, R. G. Miller, Jr., and N. U. Prabhu

Distribution of the "Blocks Adjusted for Treatments" Sum of Squares in Incomplete Block Designs — A. M. Kshirsagar

A Series of Symmetrical Group Divisible Incomplete Block Designs — D. A. Sprott

Alternative Proof of a Theorem of Birnbaum and Pyke — Nicolaas H. Kuiper

Quasi-Ranges of Samples from an Exponential Population — Paul R. Rider

Acknowledgment of Priority — Ralph G. Stanton

Correction to "Random Orthogonal Transformations and their Use in Some Classical Distribution Problems in Multivariate Analysis" — Robert A. Wijsman

Abstracts of Papers

News and Notices

Report of the Monterey, California, Meeting

Final Editorial Report

Publications Received

Address orders for subscriptions and back numbers to

A. H. BOWKER, *Treasurer*
DEPARTMENT OF STATISTICS,
STANFORD UNIVERSITY
Stanford, California

AUTHOR INDEX

(Continued from cover 2)

Dostupov, B. G.-Pougatchev, V. S. 1087
Douglas, J., Jr. 1060
Doyle, T. C. 1102
Drimi, M.-Hanš, O. 1086
Duffin, R. J.-Schild, A. 1101
*Dugué, D. 1086
Dugundji, J. 1070
Duimio, F. See Albertoni, S.
Dumitrescu, D.-Ionescu, V.-Toth, R. 1083
Dunham, B. 1147
Dutt, S. B. 1105, 1106
Dwinger, P. 1034
Dyer, E. 1071
Easterfield, T. E. 1066
Edwards, S. F. 1133
Efimov, A. V. 1049
Ehrmann, H. 1054
Eimer, C. 1111
*Eirich, F. R. (Editor). 1112
Elldin, A. 1093
Elliott, J. P. 1136
Erdös, P. 1039, 1045
Erdös, P.-Taylor, S. J. 1050
Eyges, L. 1130
Fabian, V.-Špaček, A. 1097
Falk, S. 1081
Faure, R. 1053
Feinberg, G. 1132
Fejes Tóth, L. 1074
Fekete, M.-Walsh, J. L. 1045
Feld, B. T. 1137
Feldbaum, A. A. 1149
Feller, W. 1052
Fereday, F. See Brown, R. L.
Feuerstein, E. 1126
Ficken, F. A.-Fleishman, B. A. 1059
Fikhtengol'ts, I. G. 1138
Filimonov, G. F.-Shirokov, Iu. M. 1133
Filippov, G. F. See Davydov, A. S.
*Finzi, B.-Udeschini, P. 1099
Fladt, K. 1150
Fleishman, B. A. See Ficken, F. A.
Fogagnolo Massaglia, B. 1145
Forsythe, G. E. 1079
Fortak, H. 1145
Fortet, R. M. 1089
Fox, C. 1051
Frank-Kamenetskii, D. A. 1143
Freire, R. 1081
Friedberg, R. M. 1085
Friedman, A. 1054
Friedman, B. A. See Miller, G. A.
Friedman, M. See Sheldon, J. W.
Fujita, H. See Kato, T.
Fukami, T. See Ishihara, S.
Gabriel, R. F. 1045
Gaier, D. 1048
Galkin, V. S. 1102
Gani, J. 1092
Gasapina, U. 1034
Gasparini Cattaneo, I. 1076
Gerjuoy, E. 1129
Ghizzetti, A. 1061
Gillette, D. 1147
Giovanelli, R. G. 1125
Gispert, H.-G. 1121
Giuliano, S. 1146
Givens, W. 1081
Gjellestad, G. 1143
Godeaux, L. 1078
Gold, L. 1094
Goldsmith, W. 1111
Golubenkov, V. N. 1127
Gomshor, H. 1066
Goodman, L. A. 1090
Goodman, N. R. 1098
Gordon, H.-Lorch, E. R. 1042
Gorukhov, N. F. 1111
Görtler, H. 1119
Gouarné, R. 1081, 1085
Gouyon, R. 1118
Grauert, H.-Remmert, R. 1076, 1077
Gravett, K. A. H. 1037
Gray, E. P. See Hart, R. W.
Graybill, F. A.-Marsaglia, G. .. 1095
Green, A. E.-Radok, J. R. M.-Rivlin, R. S. 1116
Greenberg, H. J. See Dorn, W. S.
Gregorini, G. 1049
Greville, T. N. E. 1079
de Groot, J. See Dekker, T. J.
de Groot, J.-de Vries, H. 1073
Grosjean, C. C. 1125
Gross, E. P.-Jackson, E. A.-Ziering, S. 1101
Grün, O. 1036
Grunsky, H. 1082
Guha, C. R.-Yih, Chia-Shun. .. 1119
Guier, W. H. See Hart, R. W.
Guilinger, W. H.-Saibel, E. ... 1119
Günther, W. 1107
Gurshi, R. N. 1102
Gustafson, W. A.-v. Krzywoblocki, M. Z. 1120
Hack, M. N. 1132
Hadwiger, H. 1073
Hajnal, A. 1031
Haldane, J. B. S. 1150
Hall, D.-Wightman, A. S. 1134
Hallman, T. M. See Siegel, R.
Hammad, H. Y. 1108
Handbook of supersonic aerodynamics. 1121
Handelman, G. See Cohen, H.
Hannan, E. J. 1098
Hanner, O. 1069
Hanš, O. See Drimi, M.
Hansen, H. H. 1071
Hardiman, N. J. 1104
Hart, R. W.-Gray, E. P.-Guier, W. H. 1135
Hartman, S.-Hulanicki, A. ... 1063
Hartree, D. R. 1137
Hasimoto, H. 1119
Haythornthwaite, R. M.-Shield, R. T. 1115
Hazelwood, R. N. See Peck, L. G.
Heigl, F. 1082
Heitler, W. 1132
Heller, J. 1080
Hermes, H. See Behnke, H.
Herstein, I. N. 1035
Herz, J.-C. 1034
Hestenes, M. R. 1080
Higgins, T. J. See Sabroff, R. R.
Hill, R. 1103, 1114
Hirzebruch, F.-Kodaira, K. .. 1077
Hodge, P. G., Jr. See Venkatraman, B.
Horwits, L. P. 1137
House, R. N., Jr. See Shaffer, B. W.
Hu, Hai-chang. 1111
Huber, A. 1046
Hudimoto, H. 1094
Hue, J. See Bernard, M.-Y.
Hulanicki, A. 1063
Hulanicki, A. See Hartman, S.
Hult, J. A. H. 1114
Hunt, G. A. 1089
Husein-Zade, M. I. 1112
Ignaczak, J. 1116
Ikuta, T. 1037
Imamura, T. 1133
Infeld, L. 1140
Ionescu, V. See Dumitrescu, D.
Ionescu Tulcea, C. T. 1064
Iordanskiĭ, S. V. 1122
Ishihara, S.-Fukami, T. 1076
Isii, K. 1086
Isiwata, T. 1069
Itô, N.-Ôhara, A. 1037
Ivanov, C. I. 1105
Iwiński, T.-Nowinski, J. 1109
Jackson, E. A. See Gross, E. P.
Jacob, H. G. 1033
Jacobs, K. 1089
Jain, M. K. 1105
Jakimovski, A. 1048
Javorskaya, I. M. 1143
Jeenel, J. 1085
Jeffreys, H.-Vicente, R. O. ... 1144
Jenkins, G. M.-Priestley, M. B. 1097
Joga Rao, C. V. 1107
Jogdeo, S. S.-Chaudhuri, A. K. 1092
Johnson, M. W.-Reissner, E. .. 1109
Joseph, V. 1139
Jung, H. 1127
Jurchescu, M. 1044
Kacner, A. 1107
Kadison, R. V.-Singer, I. M. .. 1066
Kahane, J.-P. 1050
Kalish, D. See Montague, R.
van Kampen, N. G. 1129
Kaplan, W. 1044
Karagodin, V. M. 1100
Karlin, S. 1147
Kato, T. 1068
Kato, T. See Saitô, N.
Kato, T.-Fujita, H.-Nakata, Y.-Newman, M. 1109
Kaul, R. N. 1074
*Kauzmann, W. 1135
Kavanagh, R. J. 1150
Keiner, H. 1064
Kelley, J. E., Jr. 1079
Kelly, P. J. 1073
Kemp, N. H. 1123
Kendall, D. G. 1088, 1092
*Kendall, M. G. 1093
*Kendall, M. G.-Buckland, W. R. 1093
Keogh, F. R. 1050
Keogh, F. R.-Petersen, G. M. . 1049
Kerstan, J. 1069
Khaskind, M. D. 1122
*Khinchin, A. I. 1148
Kidder, R. E. 1123
Kiefer, J. 1097
Kikuta, T. 1136
Kirzhnits, D. A. 1130, 1133
Klee, V. L., Jr. 1065
Klein, M. M. See Borowitz, S.
Klein, O. 1132
van Klinken, J. 1145
Kloos, J.-Turner, F. 1096
Koch, R. J. 1064
*Kochendörffer, R. 1033
Kočina, N. N. 1121
Kodaira, K. See Hirzebruch, F.
Kogan, A. 1121
Kohn, J. J. 1062
König, H. 1045
Konijn, H. S. 1087
Kothari, D. S. See Auluck, F. C.
Koval', P. I. 1083
Krabbe, G. L. 1067
Kraichnan, R. H. 1119
Kramer, H. P. 1054
Krasovskiĭ, N. N. 1055
v. Krzywoblocki, M. Z. See Gustafson, W. A.
v. Krzywoblocki, M. Z.-Whittenbury, C. G. 1117
Kubansky, P. N. 1122
Kuntzmann, J. 1079
Kurepa, D. 1071
Kurth, R. 1101, 1141
*Kurth, R. 1141
Kuškov, N. N. 1053
Kusukawa, K. 1120
Ladányi, K. 1137
Ladyženskaya, O. A. 1057
Lampariello, G. 1124, 1139
Lashof, R. K. 1064
Lasota, A. 1060
Lauffer, R. 1071
Lautman, D. A. 1143
Lauwerier, H. A. 1144
Lebowitz, J. L. 1099
Lees, L. See Yang, Hsun-Tiao.
Lehmann, H.-Symanzik, K.-Zimmermann, W. 1133
Leipnik, R. B. 1095
Leitmann, G. 1101
Leitner, A. See Wells, C.P.
*Le Lionnais, F. 1148
Lenard, A. 1138
Lenz, H. 1125
Leonard, D. J.-Yen, J. L. 1126
Lepropre, M. M. See Camps, F.
Levi, F. W. 1036, 1150
Lewandowski, A. 1101
Lewis, D. J. 1040
Liberman, Yu. M. 1115
Lidov, M. L. 1143
Lieberstein, H. M. 1058
*Liepmann, H. W.-Roshko, A. 1121
*Lighthill, M. J. 1066
Linfoot, E. H. 1148
Ling, Chih-Bing. 1104
Lingenberg, R. 1072
Linnik, Iu. V. See Zinger, A. A.
Linnik, Yu. V. 1096
Lipkin, H. J. 1134
Lohne, J. 1033
Lomnicki, Z. A.-Zaremba, S. K. 1090, 1098
Long, C. T. 1038
Lonsdale, K. 1117
Lorch, E. R. See Gordon, H.
Lord, F. M. 1095
Louck, J. D. 1137
Lowen, H. See Back, F. G.
Ludwig, G.-Rollnik, H. 1149
Lunc, M. 1102
*MacCamy, R. C. 1118
*McConnell, A. J. 1074
McDowell, E. L. 1116
McDowell, E. L.-Sternberg, F. 1116
McIlroy, W. See Neuringer, J. L.
McLennan, J. A., Jr. 1129
Macintyre, A. J. 1043
Mack, C. 1092
Mack, C.-Murphy, T.-Webb, N. L. 1092
Maes, S.-Amat, G. 1136
Mahalingam, S. 1101
Maharadze, L. M. 1062, 1063
Makai, E. 1051
Makar, R. H.-Mikhail, M. N. .. 1046
Mallick, D. D. 1118
Malvern, L. E. 1110
Mandel, J. 1110
Maranda, J.-M. 1035
March, N. H. See Young, W. H.
Marek, J. M. 1079
Marsaglia, G. See Graybill, F. A.
Maslen, S. H. 1117
Massera, J. L.-Schäffer, J. J. .. 1047
Masuda, K. 1040
Matschinski, M. 1071, 1072
Matthes, K. 1042
Matthews, P. T. 1138
Mauldon, J. G. 1089
Maurer-Tison, F. 1141
Mayeda, W.-Seshu, S. 1126
Mazo, R. M.-Zemach A. C. 1137
Meerov, M. V. 1149
Meisling, T. 1092
Meligy, A. S. 1131
Menger, K. 1043
Merli, L. 1051
Meschkowski, H. 1046
Meškov, A. I. 1105
Michle, W. 1147
Mikhail, M. N. 1046
Mikhail, M. N. See Makar, R. H.
Mikhailov, G. K. 1123
Mikołajska, Z. 1053
Miller, G. A.-Friedman, E. A. 1148
*Miller, K. S. 1033
Milnor, J. 1070
Milnor, J.-Shapley, L. S. 1147
Miranda, C. 1056
Miranker, W. L. 1056
Mirsky, L. 1034
Mitra, S. K. 1080
Mitra, Sujit Kumar. See Roy, Jogabrata.
Mitrinovitch, D. S. 1037
Miyake, A. 1102, 1112
Mizumoto, H. 1044
Mohan, Madan 1104
Molčanov, A. M. 1086
Moliner, F. G.-Simons, S. 1102
Montague, R.-Kalish, D. 1032
Montaldi, E. See Bocchieri, P.
Mordell, L. J. 1039
Morgan, G. W.-Pipkin, A. C.-Warner, W. H. 1119
Morita, M.-Morita, R. S. 1130
Morita, R. S. See Morita, M.
Morita, T. 1117
Morozov, N. F. 1108
Morse, M.-Transue, W. 1042
Moser, L.-Wyman, M. 1039
Moses, H. E. 1130
Motchane, L. 1069
Mrówka, S. 1064, 1068, 1069
Mrówka, S. See Balasiński, W.
Murphy, T. See Mack, C.
Murthy, S. N. B. 1121
Muštari, H. M. 1107
Nachbin, L. 1150
Nagell, T. 1062
Nahon, F. 1142
Nakada, O. 1112
Nakamura, M. 1035
Nakano, T. 1132
Nakata, Y. See Kato, T.
Nalwaskiewicz, J. 1110
Narasimhan, M. S. 1057
Nariboli, G. A. 1107
Narodetsky, M. Z. 1103
Nasr, S. K. 1085

(Continued on cover 4)

AUTHOR INDEX

(Continued from cover 3)

Naumova, L. G. 1119
Nef, W. 1065
Nesbet, R. K. 1135
de Neufville, A. 1118
Neumann, J. See Rushton, S.
Neuringer, J. L.-McIlroy, W. 1123
Nevanlinna, F. 1067
Newman, E.-Bergmann, P. G. 1139
Newman, M. See Kato, T.
Nicolovius, R. 1103
Nikodým, O. M.-Nikodým, S. 1031
Nikodým, S. See Nikodým, O. M.
Nolfi, P. 1147
Nowacki, W. 1103, 1116
Nowakowski, W. 1100
Nowiński, J. 1109
Nowiński, J. See Iwiński, T.
Nozières, P.-Pines, D. 1128
Odiot, S. 1136
Oestreicher, H. L. 1122
Ogorodnikov, K. F. 1100
Ohara, A. See Itô, N.
Oikawa, K. 1045
Olkin, I. 1097
*Ollendorff, F. 1126
Omnes, R. 1133
O'Neill, R. R. 1146
Opial, Z. 1051
Ostrowski, A. M. 1034
Ostrowsky, G. M. 1148
Owen, A. R. G. See Bliss, C. I.
Ozawa, M. 1045
Paasche, I. 1038
Pachale, H. 1051
Pachner, J. 1140
Pai, S. I. 1123
Panitchpakdi, P. See Aronszajn, N.
*Papandreou, A. G. 1145
Pâquet, P. V. 1072
Paslay, P. R. 1113
Passatore, G. 1132
Paszkowski, S. 1086
Pavlov, I. G. 1110
Payne, L. E.-Weinberger, H. F. 1110
*Peck, L. G.-Hazelwood, R. N. 1146
Pelczyński, A. 1065
Pelczyński, A.-Rolewicz, S. ... 1042
Peĭtmaldĭyan, D. V. 1108
Petersen, G. M. See Keogh, F. R.
Petrova, S. G. 1113
Peyser, G. 1059
Pham Mau Quan. 1139
Pham Tan Hoang. 1139, 1140
Phan-Van-Loc. 1137
Picone, M. 1055
Pines, D. See Nozières, P.
Pini, B. 1060
Pinkham, R. 1146
Pipkin, A. C. See Morgan, G. W.
Pirani, F. A. E. 1140
Pisarenko, G. S. 1107
Plint, C. A. See Theimer, O.
Polášek, J. 1118
Polozhy, G. N. 1106
Polodíl, G. M. 1106
Pompilj, G. 1097
Pougatchev, V. S. See Dostupov, B. G.
Požarickiĭ, G. K. 1100
Prange, R. E.-Pratt, R. H. ... 1129
Pratt, R. H. See Prange, R. E.
Priestley, M. B. See Jenkins, G. M.
Prokhoda, I. G. See Akhiezer, A. I.
Proskuryakov, A. P. 1083
Pursey, D. L. 1129
Radok, J. R. M. 1113
Radok, J. R. M. See Green, A. E.
Ragab, F. M. 1047
Ramakrishnan, Alladi-Vasudevan, R. 1142
Rasiowa, H. 1032
Razumihin, B. S. 1053
Redheffer, R. M. 1041
Rees, D. 1038
Reid, W. H. 1119
Reid, W. T. 1052
Reissner, E. 1108
Reissner, E. See Johnson, M. W.
Reissner, E.-Sledd, M. B. 1108
Reitwiesner, G. W. 1084
Rektorys, K. 1100
Remmert, R. See Grauert, H.
Reynolds, G. T. See Barford, N. C.
Riabouchinsky, D. 1080
Ribenboim, P. 1035
Rider, P. R. 1150
Rihter, V. 1087
Rikitake, T. 1145
Ringleb, F. O. 1111
Rivlin, R. S. See Green, A. E.
Roberts, J. B. 1038
Robinson, A. 1032
Robson, D. S. 1097
Rodeja, E. G. 1082
Rogers, C. A.-Shephard, G. C. 1073
Rogoziński, M. 1103
Rolewicz, S. See Pełczyński, A.
Rollnik, H. See Ludwig, G.
Root, W. L. See Davenport, W. B., Jr.
Rosati, L. A. 1071
Rosculet, M. N. 1056
Rose, M. E. 1135
Roseau, M. 1111
Rosenberg, R. M. 1053
Rosenblatt, M. 1098
Roshko, A. See Liepmann, H. W.
Ross, E. W., Jr. 1114
Roy, Jogabrata-Mitra, Sujit Kumar. 1096
Roy, S. K. 1104
*Roy, S. N. 1093
Roy, Swadesh Ranjan. See Verma, Daya Nand.
Rozanov, Yu. A. 1088
Rózsa, P. 1106
Rubinowicz, A. 1126
Rumyancev, V. V. 1100
Rushton, S.-Neumann, J. 1099
Russell, D. C. 1048
Rvačev, V. L. 1105
Sabroff, R. R.-Higgins, T. J. .. 1081
Sah, Chih-Tang-Shockley, W. 1117
Saibel, E. See Guilinger, W. H.
Saitô, N.-Kato, T. 1112
Salecker, H.-Wigner, E. P. ... 1128
Saltzer, C. 1065
Saltzman, B. 1145
Samssonow, V. 1080
Šantavý, I. 1060
Savinov, G. V.-Citovič, P. A. 1149
Schaefer, H. 1104, 1107
Schäffer, J. J. See Massera, J. L.
Schäfke, F. W. 1083
Schatz, H. 1099
Scheidegger, A. E. 1090
Scherrer, W. 1074
Schiffer, M. 1046
Schild, A. See Duffin, R. J.
Schlesinger, S. 1062
Schmidt, F. H. 1118
Schmidt, W. 1041
Schmidtmayer, J. 1080
Schneider, H. 1032
Schöpf, H.-G. 1133
Schröder, J. 1083
Schulz-Arenstorff, R. 1040
Schwarz, S. 1063
Schwinger, J. 1138
Seal, K. C. 1095
Segawa, W. 1113
Seikel, G. R. See Strandhagen, A. G.
Sen, K. K. 1143
Senju, S. 1091
Seshu, S. See Mayeda, W.
Seth, B. R. 1117
Sevast'yanov, B. A. 1088
Ševčenko, K. N. 1115
Shaffer, B. W.-House, R. N., Jr. 1114
Shannon, C. E. 1084, 1148
Shapiro, J. M. 1087
Shapley, L. S. See Milnor, J.
Sharma, Brahmadev. ... 1106, 1116
Sheldon, J. W.-Zondek, B.-Friedman, M. 1083
Shephard, G. C. See Rogers, C. A.
Sherman, D. I. 1104
Shield, R. T. 1114
Shield, R. T. See Haythornthwaite, R. M.
Shields, A. 1064
Shirokov, Iu. M. See Filimonov, G. F.
Shockley, W. See Sah, Chih-Tang.
Shtrikman, S. See Aharoni, A.
Šidák, Z. 1086
Siegel, C. L. 1048
Siegel, R.-Sparrow, E. M.-Hallman, T. M. 1127
Sigtryggsson, H.-Wiin-Nielsen, A. 1145
Simons, S. See Moliner, F. G.
Simonsen, W. 1082
Singer, I. M. See Kadison, R. V.
Singh, K. P. 1140
Singh, S. N. 1127
Sitenko, A. G. See Akhiezer, A. I.
Sledd, M. B. See Reissner, E.
Slepian, D. 1087
Słowikowski, W. 1138
Smiley, M. F. 1035
Smith, H. F. 1095
Sobol, I. M. 1079
Sokolov, A. A. 1137
Solodovnikov, V. V. See Batkov, A. M.
Solyanik-Krassa, K. V. 1105
Sorkin, Yu. I. 1035
Špaček, A. See Fabian, V.
Sparrow, E. M. 1119
Sparrow, E. M. See Siegel, R.
Speranza, F. 1075
Średniawa, B. 1138
Srivastava, A. C. 1122
Stallmann, F. 1043
Steenrod, N. E. 1069
Steenrod, N. E.-Thomas, E. .. 1070
*Stegmüller, W. 1031
Steinhaus, H. 1095
Steinmann, O. 1130
Steketee, J. A. 1105
Sternberg, E. See McDowell, E. L.
Stiefel, E. L. 1080
Stippes, M. 1103
Stöhr, A. 1040
Stojaković, M. 1079
Stojanović, R. 1076
Storchi, E. 1113
Strandhagen, A. G.-Seikel, G. R. 1117
Stratonovič, R. L. 1102
Su, Buchin. 1075
Süss, W. 1075
Švec, A. 1075
Swanson, C. A. 1054
Swetharanyam, S. 1039
Swinnerton-Dyer, H. P. F. ... 1055
Symanzik, K. See Lehmann, H.
Synge, J. L. 1140
Szelągowski, F. 1104
Szeptycki, P. 1068
Tamate, O. 1103
Tambs Lyche, R. 1041
Taub, A. H. 1139
Taylor, C. E. 1110
Taylor, S. J. See Erdős, P.
Teicher, H. 1088
Temkin, A. 1135
Tenca, L. 1034
Teplitsky, E. L. 1115
Thaler, S. A. See Boxer, R.
Theimer, O.-Plint, C. A. 1126
Thionet, P. 1095
Thomas, E. See Steenrod, N. E.
Thomas, T. Y. 1114
Thomée, V. 1059
Thurston, H. A. 1033
Thybaut, A. 1071
Tietz, H. 1045
Todeschini, B. 1100
Toll, J. S. See Wong, D. Y.
Tolmačev, V. V. 1101
Toraldo di Francia, G. 1101
Torgård, I. 1142
Tortrat, A. 1093
Tosi, A. 1150
Totaro, C. 1124
Toth, R. See Dumitrescu, D.
Transue, W. See Morse, M.
Trehan, S. K. 1123, 1125
Truesdell, C. 1103
Tsuji, M. 1043, 1044
Turán, P. 1039
Turner, F. See Kloos, J.
Udeschini, P. See Finzi, B.
Unkelbach, H. 1043
Urabe, M. 1081
Uspenskij, V. A. 1032
Vaĭnštein, B. K. 1117
Valach M. 1085
Valette, G. 1075
Vandakurov, Yu. V. 1124
Van Hove, L. 1131
Van Kranendonk, J.-Van Vleck, J. H. 1117
Van Vleck, J. H. See Van Kranendonk, J.
Varini, B. 1034
Vasudevan, R. See Ramakrishnan, Alladi.
Venkatraman B.-Hodge, P. G., Jr. 1115
Ventcel', A. D. 1060
Verblunsky, S. 1150
Verma, Daya Nand-Roy, Swadesh Ranjan. 1140
Vicente, R. O. See Jeffreys, H.
Villari, G. 1054
Vincensini, P. 1074
Viola, T. 1068, 1150
Viswanathan, K. S. 1100
Vlasov, B. F. 1105
Vodička, V. 1099
Voelz, K. 1096
Volkonskiĭ, V. A. 1089
Volland, W. 1074
Volpato, M. 1041
Vranceanu, G. 1075, 1076
de Vries, H. 1069
de Vries, H. See de Groot, J.
Wallace, A. H. 1078
Wallace, E. W. 1064
Walsh, J. L. See Fekete, M.
Walter, W. 1059
Ward, J. A. 1082
Warner, W. H. See Morgan, G. W.
Wasow, W. 1060
Waugh, W. A. O'N. 1089
Webb, N. L. See Mack, C.
Weier, J. 1070, 1071
Weigand, A. 1100
Weil, A. 1078
Weinberger, H. F. 1068
Weinberger, H. F. See Payne, L. E.
Weir, C. D. 1115
Wells, C. P.-Leitner, A. 1122
Wentzel, D. G. 1124
Wergeland, H. See Bakke, F.
Weston, J. D. 1065, 1066
White, H.-Christie, L. S. 1091
Whittenbury, C. G. See v. Krzywoblocki, M. Z.
Whittle, P. 1098
Wightman, A. S. See Hall, D.
Wigner, E. P. See Salecker, H.
Wiin-Nielsen, A. See Sigtryggsson, H.
Willis, D. G. See Biot, M. A.
Wintner, A. 1051
Wolfe, P. See Baumol, W. J.
Wong, D. Y.-Toll, J. S. 1134
Wood, W. W. 1119
Wyman, M. See Moser, L.
Yacoub, K. R. 1037
Yamada, H. 1118
Yamamoto, M. 1113
Yamanashi, S. 1035
Yang, Hsun-Tiao. 1120
Yang, Hsun-Tiao-Lees, L. 1102
Yen, J. L. See Leonard, D. J.
Yih, Chia-Shun. See Guha, C. R.
Young, J. E. 1136
Young, W. H.-March, N. H. .. 1130
Zaremba, S. K. See Lomnicki, Z. A.
Želazko, W. 1063
Zemach, A. C. See Mazo, R. M.
Zhevakin, S. A. 1143
Ziering, S. See Gross, E. P.
Zimmermann, W. See Lehmann, H.
Zinger, A. A.-Linnik, Iu. V. ... 1087
Zondek, B. See Sheldon, J. W.
Zulauf, A. 1039
*Zypkin, J. S. 1150